Analysis and Fourier transform on the Heisenberg group

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Introduction

These notes are for the time being innformal documents. They are simply a help for the mathematicians who follow the series of lectures. In particular, there is neither references nor credits. The notes are supposed to follow essentially the material exposed during the lectures.

The purpose on this series of lectures is first to introduce the audience to the analysis on the Heisenberg group and in particular to Fourier analysis on it. The Heisenberg group is the simplest example of non compact non commutative Lie group. We face some difficulties due to the non commutativity of the group and in particular the fact that the Laplace operator is not elliptic but only sub-elliptic.

Chapter 1 is a short introduction analysis on the Heisenberg group. The study is a quite detailled way the Laplace operator.

Chapter 2 studies in full detailled the properties on the Schrödinger representation used as a substitute of characters in the commutative case. In fact, we study essentially its matrix in an appropriated orthonormal basis.

Chapter 3 introduces a metric space which plays the role of the frequency space in the case of $\mathbb{R}^d$. It appears as the completion as of the set $\mathbb{N}^{2d} \times \mathbb{R} \setminus \{0\}$ for an appropriated distance.

Chapter 4 defined the notion of Fourier transform in the Heisenberg group which apperas as function on the frequency space. The characterize the range of the Schwartz class by the Fourier transform.

Chapter 5 introduces the notion of tempered distributions on the frequency space which allows to define the Fourier transform for tempered distribution on the Heisenberg group. As a conclusion we give some examples of computations of Fourier transform of some tempered distribution on $\mathbb{H}^d$, in particular the Fourier transform of functions that does not depend on the vertical variable.

Chapter 1

Analysis on the Heisenberg group: an short introduction

Introduction

This chapter is devoted to present briefly some basic fact about the Heisenberg group $\mathbb{H}^d$ which is $\mathbb{R}^{2d+1}$ equipped with a non commutative product law.

In the first section, we determine the basic object to make analysis:

– the dilation which gives the right idea of what scaling and dimension are,

– invariant measure, which allows to define convolution,

– left invariant vector fields which gives the relevant differential structure and in particular allows to define a Laplace operator.

The second section is devoted to the study of the Laplace operator; we prove in particular that it is self-adjoint and that its spectrum is the non negative real line. We conclude by exhibiting different systems of semi-norms on the Schwarz space on $\mathbb{R}^{2d+1}$ which are related to the structure of the Heisenberg group and are equivalent to the classical system associted with the structure of $\mathbb{R}^{2d+1}$.

1.1 Basic definitions

Let $T^\star\mathbb{R}^d = \mathbb{R}^d \times (\mathbb{R}^d)^\star$ be the *cotangent space* of $\mathbb{R}^d$. We shall denote by $X = (x, \xi)$ (or sometimes $Y = (y, \eta)$) a generic point of $T^\star\mathbb{R}^d$ and $\langle \xi, x\rangle$ will designate the value of the one-form ξ when applied to x.

On the space $T^\star\mathbb{R}^d$, it is natural to introduce *symplectic forms* and, more generally, *symplectic geometry.* This is the goal of the following definition.

Definition 1.1.1. *We define the symplectic form σ on $T^\star\mathbb{R}^d$ to be*

$$\sigma(X, Y) \stackrel{def}{=} \langle \xi, y\rangle - \langle \eta, x\rangle.$$

Proposition 1.1.1. *The bilinear form σ is skew-symmetric and non degenerate in the sense that*

$$\big(\forall Y \in T^\star\mathbb{R}^d\,,\ \sigma(X, Y) = 0\big) \Longleftrightarrow X = 0.$$

Proof. The fact that $\sigma(X,Y) = -\sigma(Y,X)$ is obvious. Next, if for any element Y of $T^\star\mathbb{R}^d$ we have $\sigma(X,Y) = 0$, then it is in particular the case for $Y = (y,0)$ and $Y = (0,\eta)$. Therefore

$$\forall (y,\eta) \in T^\star\mathbb{R}^d\,,\ \langle \xi, y\rangle = \langle \eta, x\rangle = 0.$$

This implies that $x = 0$ and $\xi = 0$. $\square$

Now let us introduce the *Heisenberg group* $\mathbb{H}^d$.

Definition 1.1.2. *We call Heisenberg group the set* $\mathbb{H}^d$ *equipped with the product law*

$$w \cdot w' \stackrel{\text{def}}{=} \big(X + X', s + s' + 2\sigma(X,X')\big) = \big(x + x', \xi + \xi', s + s' + 2\langle \xi, x'\rangle - 2\langle \xi', x\rangle\big).$$

where $w = (X,s) = (x,\xi,s)$ *and* $w' = (X',s') = (x',\xi',s')$ *are generic elements of* $\mathbb{H}^d$.

The law is obiously a group law. Let us notice that the inverse of w for the law $\cdot$ is $-w$.

Now let us define dilation on the Heisenberg group. Dilation are in fact diagonal linear operator (for the linear structure of $\mathbb{H}^d$ seen as $\mathbb{R}^{2d+1}$. We want these dilations δ_a to be compatible with the product law in the sense that

$$\delta_a(w \cdot w') = \delta_a(w) \cdot \delta_a(w').$$

This impose that, for positive real number a

$$\delta_a(X,s) = (aX, a^2 s). \tag{1.1}$$

Let us remark that the determinant of δ_a (seen as a linear map on $\mathbb{R}^{2d+1}$) is a^{2d+2}. This leads to the following defintion

Definition 1.1.3. *We call homogeneous dimension of* $\mathbb{H}^d$ *and denote it by* Q *the integer* $2d+2$.

Let us interest to the notion of distance on $\mathbb{H}^d$. The Heisenberg group may be endowed with the Euclidean distance d_e inherited from $\mathbb{R}^{2d+1}$. However, in most applications related to $\mathbb{H}^d$, this distance d_e is not appropriate because it is not left invariant in the sense that if τ_w is the left translation τ_w defined by

$$\tau_w(w') \stackrel{\text{def}}{=} w \cdot w' \tag{1.2}$$

we do not have $d_e(\tau_w(w'), \tau_w(w'')) = d_e(w', w'')$. It is neither homogeneous with respect to the dilations introduced in (1.1), namely $d_e(\delta_a(w), \delta_a(w')$ is not equal to $d_e(w,w')$. Let us define a distance $d_{\mathbb{H}}$ which is homogenous in the sense that

$$d_{\mathbb{H}}(\delta_a(w), \delta_a(w')) = a d_{\mathbb{H}}(w,w').$$

Definition 1.1.4. *We define*

$$d_{\mathbb{H}}(w \cdot w') \stackrel{def}{=} \rho(w^{-1} \cdot w') \quad \textit{with} \quad \stackrel{def}{=} \rho(X,s) \stackrel{def}{=} \big(|X|^4 + s^2\big)^{\frac14} = \big|\,|X|^2 \pm is\big|^{\frac12}.$$

Proposition 1.1.2. *The function d defined by (1.1.4) is a distance on* $\mathbb{H}^d$ *which is*

– *homogeneous of degree* 1*:*

$$\forall a > 0,\ \forall (w,w') \in \mathbb{H}^d \times \mathbb{H}^d,\ d(\delta_a w, \delta_a w') = a d(w,w'); \tag{1.3}$$

– *invariant by left translation:*

$$\forall (w,w',\widetilde{w})\in(\mathbb{H}^d)^3,\ d\left(\widetilde{w}\cdot w,\widetilde{w}\cdot w'\right)=d(w,w'). \tag{1.4}$$

Proof. Left invariance and homogeneity properties being obvious, let us concentrate on the triangle inequality. As

$$d_{\mathbb{H}}(w_1,w_2)=\rho(w_1^{-1}w_2)=\rho(w_1^{-1}w_3w_3^{-1}w_2)$$

the proof of the triangle inequality reduces to the proof of

$$\forall (w,w')\in\mathbb{H}^d\times\mathbb{H}^d,\ \rho(w\cdot w')\leq\rho(w)+\rho(w'). \tag{1.5}$$

We observe that

$$\begin{aligned}\rho^2(w\cdot w') &= \rho^2\big(X+X',s+s'+2\sigma(X,X')\big)\\ &= \big||X+X'|^2+i\big(s+s'+2\sigma(X,X')\big)\big|.\end{aligned}$$

As $|X+X'|^2=|X|^2+2(X\cdot X')+|X'|^2$, we get that

$$\rho^2(w\cdot w')=\big|(|X|^2+is)+(|X'|^2+is')+2(X\cdot X')+2i\sigma(X,X')\big|.$$

The triangle inequality for complex number implies that

$$\rho^2(w\cdot w')\leq\rho^2(w)+\rho^2(w')+2|(X\cdot X')|+2|\sigma(X,X')|.$$

As we have

$$\begin{aligned}\big|(X\cdot X')+i\sigma(X,X')\big| &\leq |(X\cdot X')|+|\sigma(X,X')|\\ &\leq |x|\,|x'|+|\xi|\,|\xi'|+|\xi|\,|x'|+|\xi'|\,|x|\\ &\leq |X|\,|X'|\leq\rho(w)\rho(w')\end{aligned}$$

we get Inequality (1.5) and thus the result is proved. □

Let us point out that this distance $d_{\mathbb{H}}$ is uniformely equivalent to the euclidian distance denoted by d_e. More precisely we have

Proposition 1.1.3. *We have, for any (w,w') in $\mathbb{H}^d\times\mathbb{H}^d$,*

$$\begin{aligned}d_{\mathbb{H}}(w,w') &\leq d_e(w,w')+\min\{\langle X\rangle,\langle X'\rangle\}d_e^{\frac{1}{2}}(w,w')\quad\textit{and}\\ d_e(w,w') &\leq d_{\mathbb{H}}(w,w')+2\min\{\langle X\rangle,\langle|X'|_e\rangle\}d_{\mathbb{H}}(w,w').\end{aligned}$$

Proof. Using that $\sigma(X,X')=\sigma(X,X'-X)$, let us write that

$$\begin{aligned}d_{\mathbb{H}}^2(w,w') &\leq |X-X'|^2+|s-s'-2\sigma(X,X')|\\ &\leq |X-X'|^2+|s-s'|+2|\sigma(X,X')|\\ &\leq |X-X'|^2+|s-s'|+2|X|d_e(X-X').\end{aligned}$$

Using that $\sqrt{(1+x)}\leq 1+\dfrac{x}{2}$ for non negative x we infer the first inequality by symmetry. To prove the second one, let us write that

$$|s-s'|\leq|s-s'-2\sigma(X,X')|+2|\sigma(X,X'-X)|\leq d_{\mathbb{H}}(w,w')^2+2|X|\,|X-X'|.$$

Thus, we infer that

$$d_\varepsilon(w,w')\leq d_{\mathbb{H}}(w,w')^2+2|X|\,|X-X'|+2\langle X\rangle d_{\mathbb{H}}(w,w')$$

and again conclude the proof by symmetry. □

Once we have a left invariant distance, it is natural to look for a left invariant measure. A general result claims that it exists for any locally compact group, such a measure exists and moreover it is unique up to a normalization constant. Here once we observe that the translation τ_w (which is a linear map on $\mathbb{R}^{2d+1}$ preserves the Lebesgue measure because its determinant is 1, we conclude that the Lebesgue measure is the left invariant measure on $\mathbb{H}^d$.

Once we have a left invariant measure, we can define the convolution of two integrable functions.

Definition 1.1.5. *For any two functions f and g of L^1, we define the convolution product $f\star g$ of f and g by*

$$f\star g(w) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w\cdot v^{-1})g(v)\,dv = \int_{\mathbb{H}^d} f(v)g(v^{-1}\cdot w)\,dv.$$

Let us first write the convolution in a more detailled way. By definition of the product, we have

$$\begin{aligned} f\star g(Y,s) &= \int_{\mathbb{H}^d} f\big(Y-Y', s-s'-2\sigma(Y,Y')\big)g(Y',s')dY'ds' \\ &= \int_{\mathbb{H}^d} f(Y',s')g\big(Y-Y', s-s'+2\sigma(Y,Y')\big)dY'ds'. \end{aligned} \tag{1.6}$$

As in the Euclidean case, the convolution product is an *associative* binary operation on the set of integrable functions. However, it is no longer commutative. Although the convolution product is non-commutative on $\mathbb{H}^d$, the following Young inequalities are available:

$$\|f\star g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}, \quad \text{whenever } 1\leq p,q,r\leq\infty \ \text{ and } \ \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1. \tag{1.7}$$

For a proof of this very classical result, we refer for instance to Chapter 1 of the book by H. Bahouri, J.-Y. Chemin and R. Danchin: *Fourier Analysis and Nonlinear Partial Differential Equations,* Grundlehren der mathematischen Wissenshaften, **343**, Springer, 2011.

The following *approximation of identity* result will provide us with an explicit device to approximate L^p functions with finite p, by smooth functions.

Lemma 1.1.1. *Let χ be a function of $\mathcal{D}(\mathbb{R})$ such that*

$$\int_{\mathbb{H}^d} \chi(\rho(w))\,dw = 1. \tag{1.8}$$

For $\varepsilon>0$, we denote by χ_ε the function

$$\chi_\varepsilon(w) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^Q}\chi(\rho(\delta_{\varepsilon^{-1}}w)).$$

Then we have for any p in $[1,\infty[$ and function u in L^p,

$$\lim_{\varepsilon\to0}\chi_\varepsilon\star u = \lim_{\varepsilon\to0} u\star\chi_\varepsilon = u \quad \textit{in} \quad L^p.$$

Proof. Because of Young's inequality (1.7), it is enough to prove the result for u in $\mathcal{C}_c(\mathbb{H}^d)$

which is dense in L^p for finite p. Let us write that, by virtue of (1.8),

$$\begin{aligned}(\chi_\varepsilon \star u)(w) - u(w) &= \int_{\mathbb{H}^d} \chi_\varepsilon(v)\big(u(v^{-1}\cdot w) - u(w)\big)\, dv\\ &= \int_{\mathbb{H}^d} \chi(\rho(v))\big(u(\delta_\varepsilon(v^{-1})\cdot w) - u(w)\big)\, dv,\\ (u\star \chi_\varepsilon)(w) - u(w) &= \int_{\mathbb{H}^d} \big(u(w\cdot v^{-1}) - u(w)\big)\chi_\varepsilon(v)\, dv\\ &= \int_{\mathbb{H}^d} \chi(\rho(v))\big(u(\delta_\varepsilon(v^{-1})\cdot w) - u(w)\big)\, dv.\end{aligned}$$

Because χ is compactly supported, and u is continuous and compactly supported, we get the result. $\square$

As in the classical Euclidean space, one may establish *refined Young inequalities* Inequality(refined Young) involving *weak Lebesgue spaces* defined as follows :

Definition 1.1.6. *For any q in $[1,+\infty[$ the* weak L^q space L^q_w *stands for the set of measurable functions g over $\mathbb{H}^d$ such that*

$$\|g\|^q_{L^q_w(\mathbb{H}^d)} \stackrel{def}{=} \sup_{\lambda>0} \lambda^q\big|(g| > \lambda)\big| < \infty.$$

Remark 1.1.1. *Let us point out that, since*

$$\lambda^q\big|(|g| > \lambda)\big| \le \int_{(|g|>\lambda)} |g(w)|^q\, dw \le \|g\|^q_{L^q(\mathbb{H}^d)}, \tag{1.9}$$

any function in $L^q(\mathbb{H}^d)$ is also in $L^q_w(\mathbb{H}^d)$ (with continuous embedding).

Theorem 1.1.1. *Let (p,q,r) be in $]1,\infty[^3$ and satisfy (1.7). A constant C exists such that, for any f in $L^p(\mathbb{H}^d)$ and any function g in L^q_w the function $f\star g$ belongs to L^r and satisfies*

$$\|f\star g\|_{L^r} \le C\|f\|_{L^p}\|g\|_{L^q_w}.$$

The proof is made in details for instance in Chapter 1 of the previously cited book. Theorem 1.1.1 readily implies the following *Hardy-Littlewood-Sobolev inequalities* on $\mathbb{H}^d$:

Theorem 1.1.2. *Let α in $]0,N[$, where $N = 2d+2$ is the* homogeneous dimension *of $\mathbb{H}^d$ and (p,r) in $]1,\infty[^2$ satisfy*

$$\frac{1}{p} + \frac{\alpha}{N} = 1 + \frac{1}{r}\cdot \tag{1.10}$$

Then a constant C exists such that

$$\|\rho^{-\alpha}\star f\|_{L^r} \le C\|f\|_{L^p}.$$

Proof. We can write that

$$\big\{w\,/\,\rho^{-\alpha}(w) > \lambda\big\} = \big\{w\,/\,\rho(w) < \lambda^{-\frac{1}{\alpha}}\big\}\cdot$$

Given (1.3), we thus deduce that

$$\lambda^{\frac{N}{\alpha}}\Big|\big\{w\,/\,\rho^{-\alpha}(w) > \lambda\big\}\Big| = \big|\big\{w\,/\,\rho^{-\alpha}(w) > 1\big\}\big|$$

Therefore $\rho^{-\alpha}$ belongs to $L^{\frac{N}{\alpha}}_w$ and the desired convolution inequality readily stems from Theorem 1.1.1. $\square$

It is now natural to search for the left invariant vector fields. which will play the same role as constant coefficients vector fields in the Euclidean case.

Definition 1.1.7. *A vector field $\mathcal{X}$ on $\mathbb{H}^d$ is* left invariant *if it commutes with any left translation $\tau_w(w') \stackrel{\text{def}}{=} w \cdot w'$ which means*

$$\forall w \in \mathbb{H}^d\,,\ \forall f \in C^1(\mathbb{H}^d)\,,\ (\mathcal{X}\cdot f)\circ\tau_w = \mathcal{X}\cdot(f\circ\tau_w).$$

Proposition 1.1.4. *The set of left invariant vector fields on $\mathbb{H}^d$ is the $2d+1$ vectorial space generated by*

$$\mathcal{X}_j \stackrel{\text{def}}{=} \partial_{x_j} + 2\xi_j\partial_s\,,\ \Xi_j \stackrel{\text{def}}{=} \partial_{\xi_j} - 2x_j\partial_s \quad \text{and} \quad S \stackrel{\text{def}}{=} \partial_s \quad \text{for } j \text{ in } \{1,\cdots,d\}.$$

Proof. Let us fix some $C^1(\mathbb{R}^{2d+1})$ (which means for the classical notion on C^1 functions) real valued function f on $\mathbb{H}^d$. Written in terms of the differential of f, Definition 1.1.7 recasts in

$$\forall w \in \mathbb{H}^d\,,\ (Df\cdot\mathcal{X})\circ\tau_w = D(f\circ\tau_w)\cdot\mathcal{X}.$$

Because the map τ_w is linear, the chain rule implies that

$$\forall (w,w') \in \mathbb{H}^d\times\mathbb{H}^d\,,\ Df(w\cdot w')\cdot\mathcal{X}(w\cdot w') = Df(w\cdot w')\circ D\tau_w\cdot\mathcal{X}(w').$$

As this identity must be satisfied for any function f, this gives in particular, choosing $w'=0$, that

$$\forall w \in \mathbb{H}^d\,,\ \mathcal{X}(w) = D\tau_w\,\mathcal{X}(0). \tag{1.11}$$

By definition of τ_w,

$$D\tau_w(\dot x,\dot\xi,\dot s) = \big(\dot x,\dot\xi,\dot s + 2\langle\xi,\dot x\rangle - 2\langle\dot\xi,x\rangle\big),$$

which implies that

$$D\tau_w\cdot\partial_s = S\,,\ D\tau_w\cdot\partial_{x_j} = \mathcal{X}_j \quad \text{and} \quad D\tau_w\cdot\partial_{y_j} = \Xi_j.$$

The vector $\mathcal{X}(0)$ writes

$$\mathcal{X}(0) = \alpha_0\partial_s + \sum_{j=1}^d \alpha_j\partial_{x_j} + \beta_j\partial_{\xi_j}.$$

Then using (1.11) gives the expected formula for $\mathcal{X}(w)$. Proving that, conversely, any linear combination of the vector fields S, $\mathcal{X}_j$ and Ξ_j is left invariant, is left to the reader. □

Notation. In all that follows, we denote $\mathcal{P} = (\mathcal{P}_1,\cdots,\mathcal{P}_{2d})$ with $\mathcal{P}_j \stackrel{\text{def}}{=} \mathcal{X}_j$ and $\mathcal{P}_{j+d} \stackrel{\text{def}}{=} \Xi_j$ for j in $\{1,\ldots,d\}$, we set for any multi-index α in $\{1,\ldots,2d\}^k$:

$$\mathcal{P}^\alpha \stackrel{\text{def}}{=} \mathcal{P}_{\alpha_1}\ldots\mathcal{P}_{\alpha_k}. \tag{1.12}$$

Let us study the relation between let invariant derivatives and convolution.

Proposition 1.1.5. *If P is a left invariant vector field on $\mathbb{H}^d$, then we have for all smooth functions f and g with sufficient decay at infinity:*

$$P(f\star g) = f\star(P(g)).$$

Moreover, if g is even, that is $g(w^{-1}) = g(w)$ for all w in $\mathbb{H}^d$, then

$$P(f\star g) = (P(f))\star g.$$

Proof. Thanks to the classical differentiation theorem, we have

$$P(f \star g)(w) = \int_{\mathbb{H}^d} f(v) P(g(v^{-1} \cdot w)) dv.$$

As P is left invariant, we have

$$P(g(v^{-1} \cdot w)) = (Pg)(v^{-1} \cdot w),$$

which yields the first relation.

In general $f \star (P(g))$ need not be equal to $(P(f)) \star g$. Nevertheless, in the case when g is even, we have

$$\begin{aligned} (P(f) \star g)(w) &= \int_{\mathbb{H}^d} (Pf)(v) g(v^{-1} \cdot w) dv \\ &= \int_{\mathbb{H}^d} (Pf)(v) g(w^{-1} \cdot v) dv. \end{aligned}$$

An integration by parts and the fact that P is left invariant and divergence free leads to

$$(P(f) \star g)(w) = -\int_{\mathbb{H}^d} f(v)(Pg)(w^{-1} \cdot v) dv.$$

As g is even, Pg is odd. Thus we have

$$-(Pg)(w^{-1} \cdot v) = (Pg)(v^{-1} \cdot w)$$

and the proposition is proved. □

For a function f, the notation $\nabla_{\mathbb{H}} f$ designates $(\mathcal{P}_1 f, \cdots, \mathcal{P}_{2d} f)$.

Let us define the order of a differential operator (with respect to dilations).

Definition 1.1.8. *A left invariant differential operator D is said to be order k if for any C^1 function on $\mathbb{H}^d$, we have*

$$\forall a > 0\,,\ D(f \circ \delta_a) = a^k (Df) \circ \delta_a.$$

According to this definition, the operators $\mathcal{X}_j$ and Ξ_j are first order, and the operator ∂_s is second order. Let us point out that this notion of order is different from the usual one in $\mathbb{R}^d$. To some extent, it may be compared with the case of the heat operator on $\mathbb{R}^{1+d}$, where ∂_t is 'equivalent' to two space derivatives, and is thus is of order 2.

A very important fact is that we have

$$[\mathcal{X}_j, \mathcal{X}_k] = [\Xi_j, \Xi_k] = 0 \quad \text{and} \quad [\Xi_k, \mathcal{X}_j] = 4\delta_{j,k} S, \tag{1.13}$$

where $[A, B] \stackrel{\text{def}}{=} AB - BA$ denotes the *commutator* of the operators A and B. Let us emphasize that the last relation in (1.13) provides us with an example of a commutator of two differential operators of order 1, which is of order 2. In other words, in the Heisenberg group framework, we need not gain an order of differentiation by commutation. This will cause some difficulties in what follows.

1.2 The Laplace operator and the Sobolev spaces

The *Laplacian* associated to the vector fields $\mathcal{X}_j$ and Ξ_j, namely

$$\Delta_{\mathbb{H}} \stackrel{\text{def}}{=} \sum_{j=1}^{d} (\mathcal{X}_j^2 + \Xi_j^2) \tag{1.14}$$

plays a fundamental role in the Heisenberg group. It is the sum of the square of the elements of the canonical basis of left invariant differential operators of order 1. In terms of the usual derivatives, this operator writes

$$\Delta_{\mathbb{H}} f(x,\xi,s) = \Delta_{T^\star\mathbb{R}^d} f(x,\xi,s) + 2\sum_{j=1}^{d} (\xi_j \partial_{x_j} - x_j \partial_{\xi_j})\partial_s f(x,\xi,s) + 4|X|^2 \partial_s^2 f(X,s). \tag{1.15}$$

One can now define Sobolev spaces with integer exponents as follows:

Definition 1.2.1. *For any nonnegative integer k, we denote by $H^k(\mathbb{H}^d)$ the subset of functions u in $L^2(\mathbb{H}^d)$ such that for all j in $\{0,\cdots,k\}$ and α in $\{1,\cdots,2d\}^j$, the function $\mathcal{P}^\alpha u$ belongs to $L^2(\mathbb{H}^d)$.*

Proposition 1.2.1. *The space $H^k(\mathbb{H}^d)$ endowed with the inner product*

$$(u|v)_{H^k(\mathbb{H}^d)} = \sum_{j=0}^{k} \sum_{\alpha\in\{1,\cdots,2d\}^j} (\mathcal{X}^\alpha u|\mathcal{X}^\alpha v)_{L^2}$$

is a Hilbert space, and the space $\mathcal{D}(\mathbb{H}^d)$ of test functions on $\mathbb{H}^d$ (that is smooth and compactly supported functions on $\mathbb{H}^d$) is dense in $H^k(\mathbb{H}^d)$.

Proof. In order to prove that the space $H^k(\mathbb{H}^d)$ is complete, let us consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ of $H^k(\mathbb{H}^d)$. Then $(\mathcal{X}^\alpha u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $L^2(\mathbb{H}^d)$ for any α in $\{1,\cdots,2d\}^j$ and any j in $\{0,\cdots,k\}$, and thus converges to some function u^α of $L^2(\mathbb{H}^d)$. Now, for all test function φ in $\mathcal{D}(\mathbb{H}^d)$, one may write that

$$\langle \mathcal{X}^\alpha u_n, \varphi\rangle = (-1)^{|\alpha|}\langle u_n, \mathcal{X}^\alpha \varphi\rangle,$$

whence, denoting by u the limit of $(u_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{H}^d)$,

$$\lim_{n\to+\infty} \langle \mathcal{X}^\alpha u_n, \varphi\rangle = (-1)^{|\alpha|}\langle u, \mathcal{X}^\alpha \varphi\rangle = \langle \mathcal{X}^\alpha u, \varphi\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the classical duality bracket between distribution and test functions. This means that $u^\alpha = \mathcal{X}^\alpha u$. Consequently, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $H^k(\mathbb{H}^d)$.

In order to prove the density of $\mathcal{D}(\mathbb{H}^d)$ in $H^k(\mathbb{H}^d)$, we mimic the proof of the corresponding result for Sobolev spaces on $\mathbb{R}^n$. More concretely, we fix some bump function θ in $\mathcal{D}(\mathbb{R})$ with value 1 on $[-1,1]$ and set

$$u_n \stackrel{\text{def}}{=} \theta\big(\rho(\delta_{n^{-1}}\cdot)\big)\,\big(\chi_{n^{-1}} \star u\big) \quad \text{for} \quad n \geq 1,$$

where the approximation of identity $(\chi_\varepsilon)_{\varepsilon>0}$ has been defined in Lemma 1.1.1.

Next, we write $u_n - u = v_n + w_n$ with

$$v_n \stackrel{\text{def}}{=} \big(\theta\big(\rho(\delta_{n^{-1}}\cdot)\big) - 1\big)u \quad \text{and} \quad w_n \stackrel{\text{def}}{=} \theta\big(\rho(\delta_{n^{-1}}\cdot)\big)\,\big(\chi_{n^{-1}} \star u - u\big).$$

Leibniz' rule implies that

$$\mathcal{X}^\alpha v_n = \big(\theta\big(\rho(\delta_{n^{-1}}\cdot)\big) - 1\big)\mathcal{X}^\alpha u + \sum_{\substack{\beta+\gamma=\alpha\\ \beta\neq 0}} \frac{\alpha!}{\beta!\gamma!}\mathcal{X}^\beta\big(\theta\big(\rho(\delta_{n^{-1}}\cdot)\big)\,\mathcal{X}^\gamma u.$$

Lebesgue dominated convergence theorem implies that the first term tends to 0 in $L^2(\mathbb{H}^d)$, and it is clear that the sum is $\mathcal{O}(n^{-1})$ in $L^2(\mathbb{H}^d)$. Therefore $v_n \to 0$ in $L^2(\mathbb{H}^d)$. Similarly, for w^n we have,

$$\mathcal{X}^\alpha w_n = \theta\big(\rho(\delta_{n^{-1}}\cdot)\big)(\chi_{n^{-1}} \star \mathcal{X}^\alpha u - \mathcal{X}^\alpha u) + \sum_{\substack{\beta+\gamma=\alpha\\ \beta\neq 0}} \frac{\alpha!}{\beta!\gamma!}\mathcal{X}^\beta\big(\theta\big(\rho(\delta_{n^{-1}}\cdot)\big)\,\big(\chi_{n^{-1}} \star \mathcal{X}^\gamma u - \mathcal{X}^\gamma u\big).$$

According to Lemma 1.1.1, the first term tends to 0 in L^2, and, as for v_n, the sum is $\mathcal{O}(n^{-1})$ in $L^2(\mathbb{H}^d)$. As u_n is obviously in $\mathcal{D}(\mathbb{H}^d)$, this completes the proof of the density result. □

Theorem 1.2.1. *Operator $\Delta_{\mathbb{H}}$ with domain $H^2(\mathbb{H}^d)$ is self-adjoint on $L^2(\mathbb{H}^d)$, and for any u in $H^2(\mathbb{H}^d)$, we have*[1]

$$\|u\|^2_{H^2(\mathbb{H}^d)} \sim \|u\|^2_{L^2(\mathbb{H}^d)} + \|\Delta_{\mathbb{H}} u\|^2_{L^2(\mathbb{H}^d)}.$$

Proof. As $\Delta_{\mathbb{H}}$ is symmetric with domain $H^2(\mathbb{H}^d)$, we just have to prove that the domain of the adjoint operator $(\Delta_{\mathbb{H}})^\star$ is $H^2(\mathbb{H}^d)$, that is too say, for any $u \in L^2(\mathbb{H}^d)$,

$$\Big(\forall v \in H^2(\mathbb{H}^d)\,,\ (u|\Delta_{\mathbb{H}} v)_{L^2} \leq C\|v\|_{L^2}\Big) \Longrightarrow u \in H^2(\mathbb{H}^d). \tag{1.16}$$

By an omitted density argument (use Proposition 1.2.1), it amounts to proving that

$$\forall u \in H^2(\mathbb{H}^d)\,,\ \|u\|^2_{H^2(\mathbb{H}^d)} \leq C\big(\|u\|^2_{L^2} + \|\Delta_{\mathbb{H}} u\|^2_{L^2}\big). \tag{1.17}$$

By integration by parts, we have immediately that

$$\|\nabla_{\mathbb{H}^d} u\|^2_{L^2} = -(u|\Delta_{\mathbb{H}} u)_{L^2} \leq \|u\|_{L^2}\|\Delta_{\mathbb{H}} u\|_{L^2}. \tag{1.18}$$

Now we have to control the second order derivatives. This is based on the following lemma which is is the cornerstone of the theory of subelliptic operators.

Lemma 1.2.1. *For α in $\mathbb{R}$, let us define the operator Λ^α acting on smooth functions of $\mathbb{H}^d$ by*

$$\Lambda^\alpha a \stackrel{\text{def}}{=} \mathcal{F}^{-1}_{\mathbb{R}^{2d+1}}\big(|\sigma|^\alpha \mathcal{F}_{\mathbb{R}^{2d+1}} a(\xi,\eta,\sigma)\big)$$

where $\mathcal{F}_{\mathbb{R}^{2d+1}}$ stands for the standard Fourier transform on $\mathbb{R}^{2d+1}$. Then we have for any u in $\mathcal{D}(\mathbb{H}^d)$,

$$\|\Lambda^{\frac12} u\|^2_{L^2} \leq \frac{1}{4d}(-\Delta_{\mathbb{H}} u|u)_{L^2} = \frac{1}{4d}\|\nabla_{\mathbb{H}} u\|^2_{L^2}. \tag{1.19}$$

Moreover, we have

$$\|Su\|^2_{L^2} \leq \frac{1}{4d}\|\Delta_{\mathbb{H}} u\|^2_{L^2}. \tag{1.20}$$

[1]From now on, we agree that the notation $a \sim b$ means that $C^{-1}a \leq b \leq Ca$ for some harmless positive constant C.

Proof. As obviously $\|\Lambda^{\frac12}u\|_{L^2} = \|\Lambda^{-\frac12}Su\|_{L^2}$, we get, using the commutation relation (1.13),

$$\Lambda^{-\frac12}Su = \frac14\Lambda^{-\frac12}[\Xi_j,\mathcal{X}_j]u \quad\text{and thus}\quad \|\Lambda^{\frac12}u\|_{L^2}^2 = \frac{1}{16d}\sum_{j=1}^{d}\big\|\Lambda^{-\frac12}[\Xi_j,\mathcal{X}_j]u\big\|_{L^2}^2.$$

As Λ^α is symmetric on L^2 and commutes with $\mathcal{X}_j$ and Ξ_j, and as S commutes with $\mathcal{X}_j$ and Ξ_j, we get by integration by parts,

$$\begin{aligned}\|\Lambda^{-\frac12}[\Xi_j,\mathcal{X}_j]u\|_{L^2}^2 &= \big([\Xi_j,\mathcal{X}_j]u|\Lambda^{-1}[\Xi_j,\mathcal{X}_j]u\big)_{L^2}\\ &= -\big(\mathcal{X}_ju|\Lambda^{-1}[\Xi_j,\mathcal{X}_j]\Xi_ju\big)_{L^2} + \big(\Xi_ju|\Lambda^{-1}[\Xi_j,\mathcal{X}_j]\mathcal{X}_ju\big)_{L^2}.\end{aligned}$$

Using that $\Lambda^{-1}[\Xi_j,\mathcal{X}_j]$ is a bounded operator on L^2 the norm of which is less than 4, (it is indeed the Fourier multiplier $4i\mathrm{sg}\lambda$ in the Fourier space associated to s) we deduce that

$$\begin{aligned}\|\Lambda^{-\frac12}[\Xi_j,\mathcal{X}_j]u\|_{L^2}^2 &\le 8\|\mathcal{X}_ju\|_{L^2}\|\Xi_ju\|_{L^2}\\ &\le 4(\|\mathcal{X}_ju\|_{L^2}^2 + \|\Xi_ju\|_{L^2}^2).\end{aligned}$$

Thus, because $\mathcal{X}_j$ and Ξ_j are divergence free, we have, by integration by parts

$$\begin{aligned}\|\Lambda^{\frac12}u\|_{L^2}^2 &\le \frac{1}{4d}\sum_{j=1}^{2d}(-\mathcal{X}_j^2u|u)_{L^2}\\ &\le \frac{1}{4d}(-\Delta_{\mathbb{H}}u|u)_{L^2},\end{aligned}$$

which proves the first part of the lemma. Now, applying the above inequality with $u = \Lambda^{\frac12}u$. Using that $\Lambda^{\frac12}$ and $\Delta_{\mathbb{H}}$ commute, we can write

$$\begin{aligned}\|Su\|_{L^2}^2 &= \|\Lambda u\|_{L^2}^2\\ &\le -\frac{1}{4d}(\Delta_{\mathbb{H}}\Lambda^{\frac12}u|\Lambda^{\frac12}u)_{L^2}\\ &\le -\frac{1}{4d}(\Delta_{\mathbb{H}}u|\Lambda u)_{L^2}.\end{aligned}$$

Using Cauchy-Schwartz inequality, we get

$$\|Su\|_{L^2}^2 \le \frac{1}{4d}\|\Delta_{\mathbb{H}}u\|_{L^2}\|\Lambda u\|_{L^2}.$$

As $\|Su\|_{L^2}^2 = \|\Lambda u\|_{L^2}^2$ this proves the second estimate of the lemma. □

Continuation of the proof of Theorem 1.2.1. Completing the proof is essentially algebraic matter. This is contained in the following lemma.

Lemma 1.2.2. *Let us consider two divergence free vector fields $\mathcal{Z}_1$ and $\mathcal{Z}_2$ (with real value coefficients) which both commute with $[\mathcal{Z}_1,\mathcal{Z}_2]$. Then we have*

$$\|\mathcal{Z}_1\mathcal{Z}_2u\|_{L^2}^2 = \|[\mathcal{Z}_1,\mathcal{Z}_]u\|_{L^2}^2 + (\mathcal{Z}_1^2u|\mathcal{Z}_2^2u)_{L^2}. \tag{1.21}$$

Proof. Let us write that

$$\begin{aligned}\|\mathcal{Z}_1\mathcal{Z}_2u\|_{L^2}^2 &= (\mathcal{Z}_1\mathcal{Z}_2u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2}\\ &= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2} + (\mathcal{Z}_2\mathcal{Z}_1u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2}.\end{aligned}$$

From integration by parts with respect to $\mathcal{Z}_2$ and to $\mathcal{Z}_1$, we infer that

$$
\begin{aligned}
\|\mathcal{Z}_1\mathcal{Z}_2u\|_{L^2}^2 &= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2} - (\mathcal{Z}_1u|\mathcal{Z}_2\mathcal{Z}_1\mathcal{Z}_2u)_{L^2}\\
&= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2} - (\mathcal{Z}_1u|[\mathcal{Z}_2,\mathcal{Z}_1]\mathcal{Z}_2u)_{L^2} - (\mathcal{Z}_1u|\mathcal{Z}_1\mathcal{Z}_2^2u)_{L^2}\\
&= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2} - (\mathcal{Z}_1u|[\mathcal{Z}_2,\mathcal{Z}_1]\mathcal{Z}_2u)_{L^2} + (\mathcal{Z}_1^2u|\mathcal{Z}_2^2u)_{L^2}.
\end{aligned}
$$

As $\mathcal{Z}_2$ commutes with $[\mathcal{Z}_1,\mathcal{Z}_2]$, we have by integrations by parts with respect to $\mathcal{Z}_1$ and $\mathcal{Z}_2$,

$$
\begin{aligned}
([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2} - (\mathcal{Z}_1u|[\mathcal{Z}_2,\mathcal{Z}_1]\mathcal{Z}_2u)_{L^2} &= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2} + (\mathcal{Z}_2\mathcal{Z}_1u|[\mathcal{Z}_2,\mathcal{Z}_1]u)_{L^2}\\
&= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2} - (\mathcal{Z}_1\mathcal{Z}_2u|[\mathcal{Z}_2,\mathcal{Z}_1]u)_{L^2}\\
&\quad + \|[\mathcal{Z}_2,\mathcal{Z}_1]u\|_{L^2}^2.
\end{aligned}
$$

This concludes the proof of the lemma. □

Conclusion of the proof of Theorem 1.2.1. Let us apply this identity to the vector fields $\mathcal{X}_j$ or Ξ_k involved in the definition of $\|u\|^2_{\dot H^2(\mathbb{H}^d)}$. In light of the commutation properties pointed out in (1.13), we get

$$
\begin{aligned}
\|u\|^2_{\dot H^2(\mathbb{H}^d)} &= \sum_{1\leq j,k\leq d}\big(\|\mathcal{X}_j\mathcal{X}_ku\|^2_{L^2} + \|\Xi_j\mathcal{X}_ku\|^2_{L^2} + \|\mathcal{X}_j\Xi_ku\|^2_{L^2} + \|\Xi_j\Xi_ku\|^2_{L^2}\big)\\
&\leq 2d(d-1)\|Su\|^2_{L^2} + \sum_{1\leq j,k\leq d}\big((\mathcal{X}_j^2u|\mathcal{X}_k^2u)_{L^2} + (\Xi_j^2u|\mathcal{X}_k^2u)_{L^2}\\
&\qquad\qquad + (\mathcal{X}_j^2u|\Xi_k^2u)_{L^2} + (\Xi_j^2u|\Xi_k^2u)_{L^2}\big)\\
&\leq 2d(d-1)\|Su\|^2_{L^2} + \|\Delta_{\mathbb{H}}u\|^2_{L^2}.
\end{aligned}
$$

Applying Inequality (1.20) implies the theorem. □

Lemma 1.2.1 implies immediately the following corollary

Corollary 1.2.1. *The space $H^1(\mathbb{H}^d)$ is continuously included in the space $L^2(\mathbb{R}^{2d};H^{\frac12}(\mathbb{R}))$ and also in the space $H^{\frac12}_{\rm loc}(\mathbb{R}^{2d+1})$.*

Proof. The first embedding is simply the translation of Inequality (1.19). Let us write that

$$\partial_{x_j}u = \mathcal{X}_ju - 2\xi_j\partial_su \quad\text{and}\quad \partial_{\xi_j}u = \Xi_ju + 2x_j\partial_su.$$

Using again Inequality (1.19), this implies that the functions

$$-2\xi_j\partial_su \quad\text{and}\quad 2x_j\partial_su$$

belongs locally to $H^{-\frac12}(\mathbb{R}^{2d+1})$. As the functions $\mathcal{X}_ju$ and Ξ_ju are in $L^2(\mathbb{R}^{2d+1})$, the corollary is proved. □

It is possible to describe the spectrum of the self adjoint operator $-\Delta_{\mathbb{H}}$. This is the aim of the following proposition.

Theorem 1.2.2. *The spectrum of the self adjoint operator $-\Delta_{\mathbb{H}}$ is the interval $[0,+\infty[$.*

Before proving the result, let us recall that in the classical Euclidean case, the Fourier transform allows to prove very easily this result. Indeed, for any given real number α_0, let us consider a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{S}(\mathbb{R}^n)$ such that

$$\|f_n\|_{L^2} = 1 \quad\text{and}\quad \operatorname{Supp}\widehat{f}_n \subset \Big\{\xi\in\mathbb{R}^n \,/\, \big||\xi|-|\alpha_0|\big| \leq \frac1n\Big\}.$$

As we have $-\Delta e^{i\langle\xi,x\rangle} = |\xi|^2 e^{i\langle\xi,x\rangle}$, we get that

$$\begin{aligned}(2\pi)^d\|-\Delta f_n - \alpha_0^2 f_n\|_{L^2}^2 &= \int_{\mathbb{R}^n} (|\xi|^2 - |\alpha_0|^2)^2 |\widehat{f}_n(\xi)|^2 d\xi \\ &\leq \frac{1}{n^2}\biggl(2|\alpha_0| + \frac{1}{n}\biggr)^2.\end{aligned}$$

This proves that $-\Delta - \alpha_0^2 \mathrm{Id}_{\mathbb{R}^n}$ cannot have a continuous inverse and that the spectrum of $-\Delta$ thus contains the interval $[0,+\infty[$ (the converse being obvious). In fact, the Fourier transform provides us with a description of the spectral measure.

Proof of Theorem 1.2.2. Note that (1.18) implies that the spectrum of $-\Delta_{\mathbb{H}}$ is included in $[0,+\infty[$. To prove the other inclusion, introduce the following function

$$\Theta_\lambda : \left\{ \begin{array}{ccc} \mathbb{H}^d & \rightarrow & \mathbb{C} \\ (Y,s) & \longmapsto & e^{is\lambda} e^{-\lambda|Y|^2}, \end{array}\right.$$

which will play an analogous role here as the function $x \mapsto e^{-i\langle\xi,x\rangle}$ in the $\mathbb{R}^n$ case. We claim that

$$-\Delta_{\mathbb{H}}\Theta_\lambda = 4\lambda d \Theta_\lambda \tag{1.22}$$

which is the analog of $-\Delta_x e^{i\langle\xi,x\rangle} = |\xi|^2 e^{i\langle\xi,x\rangle}$. Let us check this formula. We have

$$\begin{aligned} &\mathcal{X}_j\Theta_\lambda = -2\lambda(y_j - i\eta_j)\Theta_\lambda \quad\text{and thus}\quad \mathcal{X}_j^2\Theta_\lambda = \bigl(-2\lambda + 4\lambda^2(y_j - i\eta_j)^2\bigr)\Theta_\lambda \quad\text{and}\\ &\Xi_j\Theta_\lambda = -2\lambda(\eta_j + iy_j)\Theta_\lambda \quad\text{and thus}\quad \Xi_j^2\Theta_\lambda = \bigl(-2\lambda + 4\lambda^2(\eta_j + iy_j)^2\bigr)\Theta_\lambda,\end{aligned}$$

which obviously gives Formula (1.22). Then for a given function χ of $\mathcal{D}(]0,\infty[)$, let us define the function $T\chi$ from $\mathbb{H}^d$ to $\mathbb{C}$ by

$$T\chi(Y,s) = \sqrt{2\pi}\,\pi^{-\frac{d}{2}}\mathcal{F}_R^{-1}\bigl(\chi e^{-\cdot|Y|^2}\bigr).$$

Using Fourier Plancherel theorem for the Fourier transform on $\mathbb{R}$, we get

$$\begin{aligned}\|T\chi\|_{L^2}^2 &= \pi^{-d}\int_{T^\star\mathbb{R}^d\times\mathbb{R}} \chi^2(\lambda) e^{-2\lambda|Y|^2}\lambda^d\, dY\, d\lambda \\ &= \|\chi\|_{L^2(\mathbb{R})}^2.\end{aligned}$$

Moreover, Relation (1.22) implies that

$$-\Delta_{\mathbb{H}} T\chi = T\widetilde{\chi} \quad\text{with}\quad \widetilde{\chi}(\lambda) \stackrel{\text{def}}{=} 4\lambda d\chi(\lambda). \tag{1.23}$$

Now for any non negative real number λ_0, let us consider a sequence $(\chi_n)_{n\in\mathbb{N}}$ of functions of $\mathcal{D}(]0,\infty[)$ such that

$$\int_{\mathbb{R}} \chi_n^2(\lambda)\, d\lambda = 1 \quad\text{and}\quad \operatorname{Supp}\chi_n \subset \Bigr]\lambda_0, \lambda_0 + \frac{1}{n}\Bigl[.$$

Then we have that $\|T_{\chi_n}\|_{L^2}^2 = 1$ and, using Equality (1.23),

$$\begin{aligned}\|-\Delta_{\mathbb{H}} T_{\chi_n} - 4\lambda_0 d\, T_{\chi_n}\|_{L^2(\mathbb{H}^d)}^2 &= \|T_{\widetilde{\chi}_n - 4\lambda_0\chi_n}\|_{L^2(\mathbb{R})}^2 \\ &= \int_{\mathbb{R}} 16(\lambda-\lambda_0)^2 d^2\chi_n^2(\lambda)\, d\lambda \\ &\leq \frac{16d^2}{n^2},\end{aligned}$$

and thus $-\Delta_{\mathbb{H}} - 4\lambda_0 d\,\mathrm{Id}_{\mathbb{H}^d}$ cannot have a continuous inverse. As the spectrum of a self-adjoint operator is closed subset of $\mathbb{R}$, the theorem is proved. $\square$

Let us conclude this section by a theorem which is a generalization of Theorem 1.2.1 and wich will be useful in the nex section.

Theorem 1.2.3. *For any positive integer ℓ, we have*

$$\|\Delta_{\mathbb{H}}^{\ell} u\|_{L^2}^2 \leq \|u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2 \leq C_\ell^2 \|\Delta_{\mathbb{H}}^{\ell} u\|_{L^2}^2,$$

with $C_\ell = (Cd)^\ell\, \ell!$ for some absolute constant C.

Proof. The left inequality being obvious, let us focus on the proof of the right inequality. We proceed by induction on ℓ, the case $\ell = 0$ being trivial. So let us now assume that the theorem holds for some nonnegative integer ℓ. By definition of the Sobolev semi-norms, we have, using the induction hypothesis,

$$\begin{aligned}
\|u\|_{\dot{H}^{2\ell+2}(\mathbb{H}^d)}^2 &= \sum_{1\leq j,k\leq d} \big(\|\mathcal{X}_j\mathcal{X}_k u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2 + \|\Xi_j\mathcal{X}_k u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2 \\
&\qquad\qquad + \|\mathcal{X}_j\Xi_k u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2 + \|\Xi_j\Xi_k u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2\big) \\
&\leq C_\ell \sum_{1\leq j,k\leq d} \big(\|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}_j\mathcal{X}_k u\|_{L^2}^2 + \|\Delta_{\mathbb{H}}^{\ell}\Xi_j\mathcal{X}_k u\|_{L^2}^2 + \|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}_j\Xi_k u\|_{L^2}^2 + \|\Delta_{\mathbb{H}}^{\ell}\Xi_j\Xi_k u\|_{L^2}^2\big)\cdot \qquad (1.24)
\end{aligned}$$

Now, we have to prove a generalization of Identity (1.21). Let us consider $\mathcal{X}$ and $\mathcal{Y}$ two divergence free vector fields such that $\mathcal{X}$, $\mathcal{Y}$ and $\Delta_{\mathbb{H}}$ commute with $[\mathcal{X},\mathcal{Y}]$. Omitted computations lead to

$$\|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}\mathcal{Y}u\|_{L^2}^2 = (\Delta_{\mathbb{H}}^{\ell}\mathcal{X}^2 u|\Delta_{\mathbb{H}}^{\ell}\mathcal{Y}^2 u)_{L^2} + \sum_{m=1}^{6} R_m(u), \qquad (1.25)$$

where the terms $R_m(u)$ are defined by

$$\begin{aligned}
R_1(u) &\overset{\text{def}}{=} \big(\Delta_{\mathbb{H}}^{\ell}[\mathcal{X},\mathcal{Y}]u|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}\mathcal{Y}u\big)_{L^2} \\
R_2(u) &\overset{\text{def}}{=} \big([\Delta_{\mathbb{H}}^{\ell},\mathcal{Y}]\mathcal{X}u|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}\mathcal{Y}u\big)_{L^2} \\
R_3(u) &\overset{\text{def}}{=} \big(\mathcal{Y}\Delta_{\mathbb{H}}^{\ell}\mathcal{X}u|[\Delta_{\mathbb{H}}^{\ell},\mathcal{X}]\mathcal{Y}u\big)_{L^2} \\
R_4(u) &\overset{\text{def}}{=} \big(\Delta_{\mathbb{H}}^{\ell}\mathcal{X}u|\Delta_{\mathbb{H}}^{\ell}[\mathcal{X},\mathcal{Y}]\mathcal{Y}u\big)_{L^2} \\
R_5(u) &\overset{\text{def}}{=} \big([\mathcal{X},\Delta_{\mathbb{H}}^{\ell}]\mathcal{X}u|\mathcal{Y}\Delta_{\mathbb{H}}^{\ell}\mathcal{Y}u\big)_{L^2} \\
R_6(u) &\overset{\text{def}}{=} \big(\Delta_{\mathbb{H}}^{\ell}\mathcal{X}^2 u|[\mathcal{Y},\Delta_{\mathbb{H}}^{\ell}]\mathcal{Y}u\big)_{L^2}.
\end{aligned}$$

Taking advantage the fact that

$$[\mathcal{X}_j,\Delta_{\mathbb{H}}] = -8\Xi_j S \quad \text{and} \quad [\Xi_j,\Delta_{\mathbb{H}}] = 8\mathcal{X}_j S. \qquad (1.26)$$

and of the fact that $\mathcal{X}_j$ and Ξ_j commute with S, and using that

$$[\mathcal{X}_j,\Delta_{\mathbb{H}}^{\ell}] = \sum_{m=0}^{\ell-1} \Delta_{\mathbb{H}}^{m}[\mathcal{X}_j,\Delta_{\mathbb{H}}]\Delta_{\mathbb{H}}^{\ell-m-1} \quad \text{for} \quad j = 1,\cdots,d,$$

we infer that

$$[\mathcal{X}_j,\Delta_{\mathbb{H}}^{\ell}] = -8\Big(\sum_{m=0}^{\ell-1}\Delta_{\mathbb{H}}^{m}\Xi_j\Delta_{\mathbb{H}}^{\ell-m-1}\Big)S \quad \text{and} \quad [\Xi_j,\Delta_{\mathbb{H}}^{\ell}] = 8\Big(\sum_{m=0}^{\ell-1}\Delta_{\mathbb{H}}^{m}\mathcal{X}_j\Delta_{\mathbb{H}}^{\ell-m-1}\Big)S. \qquad (1.27)$$

Thus, if $\mathcal{X}$ and $\mathcal{Y}$ belong to the family $(\mathcal{X}_j)_{1\leq j\leq 2d}$ then all the terms $R_m(u)$ may be written (using (1.13), (1.27), and performing integration by parts as the case may be) as a linear combination of terms of the type

$$(\mathcal{X}^\alpha u|\mathcal{X}^\beta Su)_{L^2} \quad \text{with} \quad (\alpha,\beta)\in\mathcal{I}^{2\ell+2}\times\mathcal{I}^{2\ell}.$$

Note that there are at most $\mathcal{O}((\ell+1)^2)$ such terms. Now, by definition of Sobolev norms, we have for (α,β) in $\mathcal{I}^{2\ell+2}\times\mathcal{I}^{2\ell}$,

$$|(\mathcal{X}^\alpha u|\mathcal{X}^\beta Su)_{L^2}|\leq \|u\|_{\dot H^{2\ell+2}(\mathbb{H}^d)}\|Su\|_{\dot H^{2\ell}(\mathbb{H}^d)}.$$

Using Inequality (1.24) and Identity (1.25), we deduce that for some universal constant C,

$$\|u\|^2_{\dot H^{2\ell+2}(\mathbb{H}^d)}\leq C_\ell\big(\|\Delta_{\mathbb{H}}^{\ell+1}u\|^2_{L^2}+Cd^2(\ell+1)^2\|u\|_{\dot H^{2\ell+2}(\mathbb{H}^d)}\|Su\|_{\dot H^{2\ell}(\mathbb{H}^d)}\big).$$

The induction hypothesis implies that

$$\|u\|^2_{\dot H^{2\ell+2}(\mathbb{H}^d)}\leq C_\ell\big(\|\Delta_{\mathbb{H}}^{\ell+1}u\|^2_{L^2}+CC_\ell d^2(\ell+1)^2\|u\|_{\dot H^{2\ell+2}(\mathbb{H}^d)}\|\Delta_{\mathbb{H}}^{\ell}Su\|_{L^2}\big).$$

As S and $\Delta_{\mathbb{H}}$ commute, we get, using Inequality (1.20)

$$\|u\|^2_{\dot H^{2\ell+2}(\mathbb{H}^d)}\leq C^2C_\ell^2 d^2(\ell+1)^2\|\Delta_{\mathbb{H}}^{\ell+1}u\|^2_{L^2}$$

and the theorem is proved. $\square$

1.3 Remarks on the Schwartz space on the Heisenberg group

We conclude this chapter with some remarks on the Schwartz space of smooth rapidly decreasing functions of $\mathbb{H}^d$. The following proposition states that the usual semi-norms on the Schwartz class and the semi-norms using the structure of $\mathbb{H}^d$ are equivalent.

Proposition 1.3.1. *Let us introduce the notations*

$$(M_{\mathbb{H}}f)(X,s)\stackrel{\text{def}}{=}(|X|^2-is)f(X,s).$$

Moreover if α is in $\mathbb{N}^{1+2d}$, we define

$$w^\alpha\stackrel{\text{def}}{=}s^{\alpha_0}x_1^{\alpha_1}\cdots x_d^{\alpha_d}\xi_1^{\alpha_{d+1}}\cdots\xi_d^{\alpha_{2d}} \quad \text{and} \quad \widetilde{|\alpha|}\stackrel{\text{def}}{=}2\alpha_0+\alpha_1\cdots+\alpha_{2d}.$$

Then the three families of semi-norms defined on $\mathcal{S}(\mathbb{H}^d)$ by

$$\begin{aligned}
\|f\|^2_{k,\mathcal{S}(\mathbb{H}^d)} &\stackrel{\text{def}}{=} \|f\|^2_{L^2}+\|M^k_{\mathbb{H}}f\|^2_{L^2}+\big\|\Delta^k_{\mathbb{H}}f\big\|^2_{L^2}\\
N_k^2(f) &\stackrel{\text{def}}{=} \sum_{\widetilde{|\alpha|}+|\beta|\leq k}\big\|w^\alpha\mathcal{X}^\beta f\big\|^2_{L^2} \quad \text{and}\\
\widetilde N_k^2(f) &\stackrel{\text{def}}{=} \sum_{\widetilde{|\alpha|}+|\beta|\leq k}\big\|\mathcal{X}^\beta(w^\alpha f)\big\|^2_{L^2} \quad \text{and}
\end{aligned}$$

are all equivalent to the classical semi norm on $\mathcal{S}(\mathbb{R}^{2d+1})$.

Proof. Observe that $\mathcal{X}^\gamma w^{\gamma'}$ is an homogeneous polynomial of degree $\gamma' - \gamma$, and equal to 0 if the length of γ is greater than the length of γ'. Thus, thanks to Leibniz' rule, we have

$$[\mathcal{X}^\beta, w^\alpha] f(w) = \sum_{\substack{\widetilde{|\alpha|} \leq \widetilde{|\alpha|} - 1 \\ |\beta'| \leq |\beta| - 1}} a_{\alpha,\alpha',\beta',\beta}\, w^{\alpha'} \mathcal{X}^{\beta'} f(w). \tag{1.28}$$

Then an omitted induction implies that

$$C_k^{-1} \widetilde{N}_k(f) \leq N_k(f) \leq C_k \widetilde{N}_k(f). \tag{1.29}$$

Let us observe that Leibniz' rule implies that, if $\widetilde{|\alpha|} + |\beta| \leq k$, then

$$\mathcal{X}^\beta (w^\alpha f)(w) = w^\alpha \mathcal{X}^\beta f(w) + \sum_{\widetilde{|\alpha'|} + |\beta'| \leq k-2} a_{\alpha',\beta'} w^{\alpha'} \mathcal{X}^{\beta'} f(w). \tag{1.30}$$

As obviously $\|f\|_k$ is less than or equal to $N_{2k}(f)$, the proof is the fact that the three semi-norms are equivalent reduces to the proof of

$$\forall k \in \mathbb{N}\,,\ \exists (C_k, M_k) \,/\, \forall f \in \mathcal{S}(\mathbb{H}^d)\,,\ N_k(f) \leq C_k \|f\|^2_{k,\mathcal{S}(\mathbb{H}^d)}. \tag{1.31}$$

Using an integration by parts, we get

$$\int_{\mathbb{H}^d} w^\alpha \mathcal{X}^\beta f(w) w^\alpha \mathcal{X}^\beta \overline{f}(w)\, dw = (-1)^{|\beta|} \int_{\mathbb{H}^d} f(w) \mathcal{X}^\beta \big(w^{2\alpha} \mathcal{X}^\beta \overline{f}(w)\big) dw$$

Applying (1.28), we get that

$$\int_{\mathbb{H}^d} f(w) \mathcal{X}^\beta \big(w^{2\alpha} \overline{\mathcal{X}}^\beta f(w)\big) dw = \sum_{\substack{\widetilde{|\alpha'|} \leq 2|\alpha| \\ |\beta'| \leq |\beta|}} a_{\alpha,\alpha',\beta,\beta'} \int_{\mathbb{H}^d} w^{\alpha'} f(w) \mathcal{X}^{\beta'} \mathcal{X}^\beta \overline{f}(w)\, dw$$

Thanks to Cauchy-Schwartz inequality and by definition of $M_{\mathbb{H}}$, we get, applying Theorem 1.2.3,

$$\begin{aligned} \sum_{\substack{\widetilde{|\alpha'|} \leq 2|\alpha| \\ |\beta'| \leq |\beta|}} a_{\alpha,\alpha',\beta,\beta'} \int_{\mathbb{H}^d} w^{\alpha'} f(w) \mathcal{X}^{\beta'} \mathcal{X}^\beta \overline{f}(w)\, dw &\lesssim \|(\mathrm{Id} + M^k_{\mathbb{H}} f\|_{L^2} \|f\|_{H^{2k}(\mathbb{H}^d)} \\ &\lesssim \|f\|^2_{L^2} + \|M^k_{\mathbb{H}} f\|^2_{L^2} + \|\Delta^k_{\mathbb{H}} f\|^2_{L^2(\mathbb{H}^d)}. \end{aligned}$$

This proves that the three families of equivalent. In order to prove that there are equivalent to the classical family, let us observe that as

$$S = \frac{1}{4}[\Xi_j, \mathcal{X}_j]\,, \partial_{x_j} = \mathcal{X}_j - \frac{y_j}{2}(\Xi_j \mathcal{X}_j - \mathcal{X}_j \Xi_j) \quad \text{and} \quad \partial_{y_j} = \Xi_j + \frac{x_j}{2}(\Xi_j \mathcal{X}_j - \mathcal{X}_j \Xi_j)$$

and, for all j in $\{1, \cdots, d\}$, we infer that

$$\widetilde{\|f\|}_{k,\mathcal{S}(\mathbb{R}^{2d+1})} \leq C \|f\|_{2k,\mathcal{S}(\mathbb{H}^d)}$$

where $\widetilde{\|f\|}^2_{k,\mathcal{S}(\mathbb{R}^{2d+1})} \stackrel{\text{def}}{=} \sum_{|\alpha|+|\beta| \leq k} \|x^\alpha \partial_\beta f(x)\|^2_{L^2(\mathbb{R}^{2d+1})}$ which ends the proof of the proposition.

□

In all that follows, we shall use indistinctly the above three different types of semi-norms, and the Schwartz class on $\mathbb{R}^{2d+1}$ (or, equivalently, on $\mathbb{H}^d$) will be sometimes just denoted by $\mathcal{S}$. When we denote $\mathcal{S}(\mathbb{H}^d)$ we have in mind to use the semi-norm related to the structure of the Heisenberg group. The only property we have to prove here concerns the convolution in the sense of the Heisenberg group.

As a conclusion, let us prove a continuity theorem about the convolution which will be useful when we shall study the Fourier transform of distribution.

Proposition 1.3.2. *Let us define $L^1(\mathbb{H}^d)$ the space of integrable functions f on $\mathbb{H}^d$ such that*

$$\forall k \in \mathbb{N}\,,\ \|f\|_{k,L^1_{\mathcal{S}}(\mathbb{H}^d)} \stackrel{\text{def}}{=} \|M^k_{\mathbb{H}} f\|_{L^1(\mathbb{H}^d)} < \infty.$$

The convolution product on $\mathbb{H}^d$ is bicontinuous operator from $L^1_{\mathcal{S}}(\mathbb{H}^d) \times \mathcal{S}(\mathbb{H}^d)$ in the sense that for any k, a constant C_k and an integer M_k exist such that

$$\|f \star g\|_{k,\mathcal{S}(\mathbb{H}^d)} \leq C_k \|f\|_{M_k,L^1_{\mathcal{S}}(\mathbb{H}^d)} \|g\|_{M_k,\mathcal{S}(\mathbb{H}^d)}.$$

Proof. Let us first observe that using Proposition 1.1.5, we have $\Delta^\ell_{\mathbb{H}}(f \star g) = f \star \Delta^\ell_H g$. Then we have to study the action of $M^k_{\mathbb{H}}$ on the convolution. This is described by the following lemma.

Lemma 1.3.1. *Let us define*

$$(M^\pm_j f)(Y,s) \stackrel{\text{def}}{=} (y_j \pm i\eta_j) f(Y,s) \quad \text{and} \quad M_0 f(s) \stackrel{\text{def}}{=} -isf(Y,s).$$

Then we have the identities

$$\begin{aligned} M^\pm_j(f \star g) &= (M^\pm_j f) \star g + f \star (M^\pm_j g) \quad \text{and} \\ M_0(f \star g) &= (M_0 f) \star g + f \star (M_0 g) - \sum_{j=1}^d \big((M^+_j f) \star (M^-_j g) - (M^-_j f) \star (M^+_j g)\big). \end{aligned}$$

Proof. Checking the first formula is obvious. As for the second one, we use that by definition of the convolution, we have

$$-is(f \star g)(w) = -is \int_{\mathbb{H}^d} f\big((Y-Y', s-s'-2\sigma(Y,Y')\big) g(Y',s')\,dw'.$$

From $-is = -i\big(s-s'-2\sigma(Y,Y')\big) - is' - 2i\sigma(Y,Y')$, we infer that

$$\begin{aligned} -is(f \star g)(w) &= \int_{\mathbb{H}^d} (M_0 f)\big((Y-Y', s-s'-2\sigma(Y,Y')\big) g(Y',s')\,dw' \\ &\qquad + \int_{\mathbb{H}^d} f\big((Y-Y', s-s'-2\sigma(Y,Y')\big)(M_0 g)(Y',s')\,dw' \\ &\qquad\quad - \int_{\mathbb{H}^d} 2i\sigma(Y,Y') f\big((Y-Y', s-s'-2\sigma(Y,Y')\big) g(Y',s')\,dw' \\ &= ((M_0 f) \star g)(w) + (f \star (M_0 g))(w) \\ &\qquad\quad - \int_{\mathbb{H}^d} 2i\sigma(Y,Y') f\big((Y-Y', s-s'-2\sigma(Y,Y')\big) g(Y',s')\,dw'\,. \end{aligned}$$

The symplectic form σ satisfies $\sigma(Y',Y')=0$. Thus by definition of σ

$$
\begin{aligned}
\Delta(f,g)(w) &\stackrel{\text{def}}{=} \int_{\mathbb{H}^d} 2i\sigma(Y,Y')f\big((Y-Y',s-s'-2\sigma(Y,Y')\big)g(Y',s')\,dw'\\
&= \int_{\mathbb{H}^d} 2i\sigma(Y-Y',Y')f\big((Y-Y',s-s'-2\sigma(Y,Y')\big)g(Y',s')\,dw'\\
&= \sum_{j=1}^d \int_{\mathbb{H}^d}\Big(2i(\eta_j-\eta'_j)f\big(Y-Y',s-s'-2\sigma(Y,Y')\big)y'_jg(Y',s')\\
&\qquad\qquad -(y_j-y'_j)f\big(Y-Y',s-s'-2\sigma(Y,Y')\big)2i\eta'_jg(Y',s')\Big)\,dw'
\end{aligned}
$$

By definition of the operator $M_j^\pm$, this gives

$$
\begin{aligned}
\Delta_j(f,g)(w) &\stackrel{\text{def}}{=} \int_{\mathbb{H}^d}\Big(2i(\eta_j-\eta'_j)f\big(Y-Y',s-s'-2\sigma(Y,Y')\big)y'_jg(Y',s')\\
&\qquad\qquad -(y_j-y'_j)f\big(Y-Y',s-s'-2\sigma(Y,Y')\big)2i\eta'_jg(Y',s')\Big)\,dw'\\
&= \frac12\big((M_j^+-M_j^-)f\big)\star\big(M_j^++M_j^-)g\big)-\frac12\big((M_j^++M_j^-)f\big)\star\big(M_j^+-M_j^-)g\big)\\
&= (M_j^+f)\star(M_j^-g)-(M_j^-f)\star(M_j^+g).
\end{aligned}
$$

This gives the result. □

Conclusion of the proof of Proposition 1.3.2 If we denote, for α in $\mathbb{N}^{2d+1}$,

$$
\mathcal{M}^\alpha \stackrel{\text{def}}{=} M_0^{\alpha_0}(M_1^+)^{\alpha_1}\cdot(M_d^+)^{\alpha_d}(M_1^-)^{\alpha_{d+1}}\cdots(M_1^-)^{\alpha_{2d}}
$$

an iterated application of Lemma 1.3.1 implies that

$$
M_{\mathbb{H}}^k\Delta_{\mathbb{H}}^k(f\star g)=\sum_{\widetilde{|}\alpha|+\widetilde{|}\alpha'|\leq 2k} a_{\alpha,\alpha'}(\mathcal{M}^\alpha f)\star(\mathcal{M}^{\alpha'}\Delta_{\mathbb{H}}^k g).
$$

Young's inequality implies that

$$
\begin{aligned}
\big\|M_{\mathbb{H}}^k\Delta_{\mathbb{H}}^\ell(f\star g)\big\|_{L^2} &\leq C_k\sum_{\widetilde{|}\alpha|+\widetilde{|}\alpha'|\leq 2k}\|\mathcal{M}^\alpha f\|_{L^1}\|\mathcal{M}^{\alpha'}\Delta_{\mathbb{H}}^k g\|_{L^2}\\
&\leq C_k\|f\|_{k,\mathcal{S}(\mathbb{H}^d)}\|g\|_{k,\mathcal{S}(\mathbb{H}^d)}.
\end{aligned}
$$

This proves the proposition. □