Analysis and Fourier transform on the Heisenberg group

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Introduction

These notes are for the time being innformal documents. They are simply a help for the mathematicians who follow the series of lectures. In particular, there is neither references nor credits. The notes are supposed to follow essentially the material exposed during the lectures.

The purpose on this series of lectures is first to introduce the audience to the analysis on the Heisenberg group and in particular to Fourier analysis on it. The Heisenberg group is the simplest example of non compact non commutative Lie group. We face some difficulties due to the non commutativity of the group and in particular the fact that the Laplace operator is not elliptic but only sub-elliptic.

Chapter 1 is a short introduction analysis on the Heisenberg group. The study is a quite detailled way the Laplace operator.

Chapter 2 studies in full detailled the properties on the Schrödinger representation used as a substitute of characters in the commutative case. In fact, we study essentially its matrix in an appropriated orthonormal basis.

Chapter 3 introduces a metric space which plays the role of the frequency space in the case of $\mathbb{R}^d$. It appears as the completion as of the set $\mathbb{N}^{2d} \times \mathbb{R} \setminus \{0\}$ for an appropriated distance.

Chapter 4 defined the notion of Fourier transform in the Heisenberg group which apperas as function on the frequency space. The characterize the range of the Schwartz class by the Fourier transform.

Chapter 5 introduces the notion of tempered distributions on the frequency space which allows to define the Fourier transform for tempered distribution on the Heisenberg group. As a conclusion we give some examples of computations of Fourier transform of some tempered distribution on $\mathbb{H}^d$, in particular the Fourier transform of functions that does not depend on the vertical variable.

Chapter 1

Analysis on the Heisenberg group: an short introduction

Introduction

This chapter is devoted to present briefly some basic fact about the Heisenberg group $\mathbb{H}^d$ which is $\mathbb{R}^{2d+1}$ equipped with a non commutative product law.

In the first section, we determine the basic object to make analysis:

- the dilation which gives the right idea of what scaling and dimension are,
- invariant measure, which allows to define convolution,
- left invariant vector fields which gives the relevant differential structure and in particular allows to define a Laplace operator.

The second section is devoted to the study of the Laplace operator; we prove in particular that it is self-adjoint and that its spectrum is the non negative real line. We conclude by exhibiting different systems of semi-norms on the Schwarz space on $\mathbb{R}^{2d+1}$ which are related to the structure of the Heisenberg group and are equivalent to the classical system associted with the structure of $\mathbb{R}^{2d+1}$.

1.1 Basic definitions

Let $T^\star\mathbb{R}^d = \mathbb{R}^d \times (\mathbb{R}^d)^\star$ be the *cotangent space* of $\mathbb{R}^d$. We shall denote by $X = (x,\xi)$ (or sometimes $Y = (y,\eta)$) a generic point of $T^\star\mathbb{R}^d$ and $\langle \xi, x\rangle$ will designate the value of the one-form ξ when applied to x.

On the space $T^\star\mathbb{R}^d$, it is natural to introduce *symplectic forms* and, more generally, *symplectic geometry.* This is the goal of the following definition.

Definition 1.1.1. *We define the symplectic form σ on $T^\star\mathbb{R}^d$ to be*

$$\sigma(X,Y) \stackrel{\text{def}}{=} \langle \xi, y\rangle - \langle \eta, x\rangle.$$

Proposition 1.1.1. *The bilinear form σ is skew-symmetric and non degenerate in the sense that*

$$\big(\forall Y \in T^\star\mathbb{R}^d\,,\ \sigma(X,Y)=0\big) \Longleftrightarrow X=0.$$

Proof. The fact that $\sigma(X,Y) = -\sigma(Y,X)$ is obvious. Next, if for any element Y of $T^\star\mathbb{R}^d$ we have $\sigma(X,Y) = 0$, then it is in particular the case for $Y = (y,0)$ and $Y = (0,\eta)$. Therefore

$$\forall (y,\eta) \in T^\star\mathbb{R}^d\,,\ \langle \xi, y\rangle = \langle \eta, x\rangle = 0.$$

This implies that $x = 0$ and $\xi = 0$. $\square$

Now let us introduce the *Heisenberg group* $\mathbb{H}^d$.

Definition 1.1.2. *We call Heisenberg group the set $\mathbb{H}^d$ equipped with the product law*

$$w \cdot w' \stackrel{\text{def}}{=} \big(X + X', s + s' + 2\sigma(X,X')\big) = \big(x + x', \xi + \xi', s + s' + 2\langle \xi, x'\rangle - 2\langle \xi', x\rangle\big).$$

where $w = (X,s) = (x,\xi,s)$ and $w' = (X',s') = (x',\xi',s')$ are generic elements of $\mathbb{H}^d$.

The law is obiously a group law. Let us notice that the inverse of w for the law $\cdot$ is $-w$.

Now let us define dilation on the Heisenberg group. Dilation are in fact diagonal linear operator (for the linear structure of $\mathbb{H}^d$ seen as $\mathbb{R}^{2d+1}$. We want these dilations δ_a to be compatible with the product law in the sense that

$$\delta_a(w \cdot w') = \delta_a(w) \cdot \delta_a(w').$$

This impose that, for positive real number a

$$\delta_a(X,s) = (aX, a^2 s). \tag{1.1}$$

Let us remark that the determinant of δ_a (seen as a linear map on $\mathbb{R}^{2d+1}$) is a^{2d+2}. This leads to the following defintion

Definition 1.1.3. *We call homogeneous dimension of $\mathbb{H}^d$ and denote it by Q the integer $2d+2$.*

Let us interest to the notion of distance on $\mathbb{H}^d$. The Heisenberg group may be endowed with the Euclidean distance d_e inherited from $\mathbb{R}^{2d+1}$. However, in most applications related to $\mathbb{H}^d$, this distance d_e is not appropriate because it is not left invariant in the sense that if τ_w is the left translation τ_w defined by

$$\tau_w(w') \stackrel{\text{def}}{=} w \cdot w' \tag{1.2}$$

we do not have $d_e(\tau_w(w'), \tau_w(w'')) = d_e(w', w'')$. It is neither homogeneous with respect to the dilations introduced in (1.1), namely $d_e(\delta_a(w), \delta_a(w')$ is not equal to $d_e(w,w')$. Let us define a distance $d_{\mathbb{H}}$ which is homogenous in the sense that

$$d_{\mathbb{H}}(\delta_a(w), \delta_a(w')) = a d_{\mathbb{H}}(w,w').$$

Definition 1.1.4. *We define*

$$d_{\mathbb{H}}(w\cdot w') \stackrel{def}{=} \rho(w^{-1}\cdot w') \quad with \quad \stackrel{def}{=} \rho(X,s) \stackrel{def}{=} \big(|X|^4 + s^2\big)^{\frac14} = \big|\,|X|^2 \pm is\big|^{\frac12}.$$

Proposition 1.1.2. *The function d defined by (1.1.4) is a distance on $\mathbb{H}^d$ which is*

– *homogeneous of degree 1:*

$$\forall a > 0,\, \forall (w,w') \in \mathbb{H}^d \times \mathbb{H}^d,\ d(\delta_a w, \delta_a w') = a d(w,w'); \tag{1.3}$$

– *invariant by left translation:*

$$\forall (w, w', \widetilde{w}) \in (\mathbb{H}^d)^3,\ d\left(\widetilde{w}\cdot w, \widetilde{w}\cdot w'\right) = d(w,w'). \tag{1.4}$$

Proof. Left invariance and homogeneity properties being obvious, let us concentrate on the triangle inequality. As

$$d_{\mathbb{H}}(w_1,w_2) = \rho(w_1^{-1}w_2) = \rho(w_1^{-1}w_3w_3^{-1}w_2)$$

the proof of the triangle inequality reduces to the proof of

$$\forall (w,w') \in \mathbb{H}^d\times\mathbb{H}^d\,,\ \rho(w\cdot w') \leq \rho(w)+\rho(w'). \tag{1.5}$$

We observe that

$$\begin{aligned} \rho^2(w\cdot w') &= \rho^2\big(X+X', s+s'+2\sigma(X,X')\big)\\ &= \big||X+X'|^2 + i\big(s+s'+2\sigma(X,X')\big)\big|. \end{aligned}$$

As $|X+X'|^2 = |X|^2 + 2(X\cdot X') + |X'|^2$, we get that

$$\rho^2(w\cdot w') = \big|(|X|^2+is) + (|X'|^2+is') + 2(X\cdot X') + 2i\sigma(X,X')\big|.$$

The triangle inequality for complex number implies that

$$\rho^2(w\cdot w') \leq \rho^2(w)+\rho^2(w') + 2|(X\cdot X')| + 2|\sigma(X,X')|.$$

As we have

$$\begin{aligned} \big|(X\cdot X') + i\sigma(X,X')\big| &\leq |(X\cdot X')| + |\sigma(X,X')|\\ &\leq |x|\,|x'| + |\xi|\,|\xi'| + |\xi|\,|x'| + |\xi'|\,|x|\\ &\leq |X|\,|X'| \leq \rho(w)\rho(w') \end{aligned}$$

we get Inequality (1.5) and thus the result is proved. □

Let us point out that this distance $d_{\mathbb{H}}$ is uniformely equivalent to the euclidian distance denoted by d_e. More precisely we have

Proposition 1.1.3. *We have, for any* (w,w') *in* $\mathbb{H}^d\times\mathbb{H}^d$,

$$\begin{aligned} d_{\mathbb{H}}(w,w') &\leq d_e(w,w') + \min\{\langle X\rangle, \langle X'\rangle\} d_e^{\frac12}(w,w') \quad \textit{and}\\ d_e(w,w') &\leq d_{\mathbb{H}}(w,w') + 2\min\{\langle X\rangle, \langle |X'|_e\rangle\} d_{\mathbb{H}}(w,w'). \end{aligned}$$

Proof. Using that $\sigma(X,X') = \sigma(X,X'-X)$, let us write that

$$\begin{aligned} d^2_{\mathbb{H}}(w,w') &\leq |X-X'|^2 + |s-s'-2\sigma(X,X')|\\ &\leq |X-X'|^2 + |s-s'| + 2|\sigma(X,X')|\\ &\leq |X-X'|^2 + |s-s'| + 2|X| d_e(X-X'). \end{aligned}$$

Using that $\sqrt{(1+x)} \leq 1+\dfrac{x}{2}$ for non negative x we infer the first inequality by symmetry. To prove the second one, let us write that

$$|s-s'| \leq |s-s'-2\sigma(X,X')| + 2|\sigma(X,X'-X)| \leq d_{\mathbb{H}}(w,w')^2 + 2|X|\,|X-X'|.$$

Thus, we infer that

$$d_\varepsilon(w,w') \leq d_{\mathbb{H}}(w,w')^2 + 2|X|\,|X-X'| + 2\langle X\rangle d_{\mathbb{H}}(w,w')$$

and again conclude the proof by symmetry. □

Once we have a left invariant distance, it is natural to look for a left invariant measure. A general result claims that it exists for any locally compact group, such a measure exists and moreover it is unique up to a normalization constant. Here once we observe that the translation τ_w (which is a linear map on $\mathbb{R}^{2d+1}$ preserves the Lebesgue measure because its determinant is 1, we conclude that the Lebesgue measure is the left invariant measure on $\mathbb{H}^d$.

Once we have a left invariant measure, we can define the convolution of two integrable functions.

Definition 1.1.5. *For any two functions f and g of L^1, we define the convolution product $f \star g$ of f and g by*

$$f \star g(w) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w \cdot v^{-1}) g(v)\, dv = \int_{\mathbb{H}^d} f(v) g(v^{-1} \cdot w)\, dv.$$

Let us first write the convolution in a more detailled way. By definition of the product, we have

$$\begin{aligned} f \star g(Y, s) &= \int_{\mathbb{H}^d} f\big(Y - Y', s - s' - 2\sigma(Y, Y')\big) g(Y', s') dY' ds' \\ &= \int_{\mathbb{H}^d} f(Y', s') g\big(Y - Y', s - s' + 2\sigma(Y, Y')\big) dY' ds'. \end{aligned} \tag{1.6}$$

As in the Euclidean case, the convolution product is an *associative* binary operation on the set of integrable functions. However, it is no longer commutative. Although the convolution product is non-commutative on $\mathbb{H}^d$, the following Young inequalities are available:

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}, \quad \text{whenever } 1 \leq p, q, r \leq \infty \ \text{ and } \ \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{1.7}$$

For a proof of this very classical result, we refer for instance to Chapter 1 of the book by H. Bahouri, J.-Y. Chemin and R. Danchin: *Fourier Analysis and Nonlinear Partial Differential Equations,* Grundlehren der mathematischen Wissenshaften, **343**, Springer, 2011.

The following *approximation of identity* result will provide us with an explicit device to approximate L^p functions with finite p, by smooth functions.

Lemma 1.1.1. *Let χ be a function of $\mathcal{D}(\mathbb{R})$ such that*

$$\int_{\mathbb{H}^d} \chi(\rho(w))\, dw = 1. \tag{1.8}$$

For $\varepsilon > 0$, we denote by χ_ε the function

$$\chi_\varepsilon(w) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^Q} \chi(\rho(\delta_{\varepsilon^{-1}} w)).$$

Then we have for any p in $[1, \infty[$ and function u in L^p,

$$\lim_{\varepsilon \to 0} \chi_\varepsilon \star u = \lim_{\varepsilon \to 0} u \star \chi_\varepsilon = u \quad \textit{in} \quad L^p.$$

Proof. Because of Young's inequality (1.7), it is enough to prove the result for u in $\mathcal{C}_c(\mathbb{H}^d)$

which is dense in L^p for finite p. Let us write that, by virtue of (1.8),

$$\begin{aligned}(\chi_\varepsilon \star u)(w) - u(w) &= \int_{\mathbb{H}^d} \chi_\varepsilon(v)\bigl(u(v^{-1}\cdot w) - u(w)\bigr)\,dv\\ &= \int_{\mathbb{H}^d} \chi(\rho(v))\bigl(u(\delta_\varepsilon(v^{-1})\cdot w) - u(w)\bigr)\,dv,\\ (u\star\chi_\varepsilon)(w) - u(w) &= \int_{\mathbb{H}^d} \bigl(u(w\cdot v^{-1}) - u(w)\bigr)\chi_\varepsilon(v)\,dv\\ &= \int_{\mathbb{H}^d} \chi(\rho(v))\bigl(u(\delta_\varepsilon(v^{-1})\cdot w) - u(w)\bigr)\,dv.\end{aligned}$$

Because χ is compactly supported, and u is continuous and compactly supported, we get the result. □

As in the classical Euclidean space, one may establish *refined Young inequalities* Inequality(refined Young) involving *weak Lebesgue spaces* defined as follows :

Definition 1.1.6. *For any q in $[1,+\infty[$ the* weak L^q space L^q_w *stands for the set of measurable functions g over $\mathbb{H}^d$ such that*

$$\|g\|^q_{L^q_w(\mathbb{H}^d)} \stackrel{def}{=} \sup_{\lambda>0} \lambda^q\bigl|(g|>\lambda)\bigr| < \infty.$$

Remark 1.1.1. *Let us point out that, since*

$$\lambda^q\bigl|(|g|>\lambda)\bigr| \le \int_{(|g|>\lambda)} |g(w)|^q\,dw \le \|g\|^q_{L^q(\mathbb{H}^d)}, \tag{1.9}$$

any function in $L^q(\mathbb{H}^d)$ is also in $L^q_w(\mathbb{H}^d)$ (with continuous embedding).

Theorem 1.1.1. *Let (p,q,r) be in $]1,\infty[^3$ and satisfy (1.7). A constant C exists such that, for any f in $L^p(\mathbb{H}^d)$ and any function g in L^q_w the function $f\star g$ belongs to L^r and satisfies*

$$\|f\star g\|_{L^r} \le C\|f\|_{L^p}\|g\|_{L^q_w}.$$

The proof is made in details for instance in Chapter 1 of the previously cited book. Theorem 1.1.1 readily implies the following *Hardy-Littlewood-Sobolev inequalities* on $\mathbb{H}^d$:

Theorem 1.1.2. *Let α in $]0,N[$, where $N=2d+2$ is the* homogeneous dimension *of $\mathbb{H}^d$ and (p,r) in $]1,\infty[^2$ satisfy*

$$\frac{1}{p}+\frac{\alpha}{N} = 1+\frac{1}{r}\cdot \tag{1.10}$$

Then a constant C exists such that

$$\|\rho^{-\alpha}\star f\|_{L^r} \le C\|f\|_{L^p}.$$

Proof. We can write that

$$\bigl\{w\,/\,\rho^{-\alpha}(w)>\lambda\bigr\} = \bigl\{w\,/\,\rho(w)<\lambda^{-\frac{1}{\alpha}}\bigr\}\cdot$$

Given (1.3), we thus deduce that

$$\lambda^{\frac{N}{\alpha}}\Bigl|\bigl\{w\,/\,\rho^{-\alpha}(w)>\lambda\bigr\}\Bigr| = \bigl|\bigl\{w\,/\,\rho^{-\alpha}(w)>1\bigr\}\bigr|$$

Therefore $\rho^{-\alpha}$ belongs to $L^{\frac{N}{\alpha}}_w$ and the desired convolution inequality readily stems from Theorem 1.1.1. □

It is now natural to search for the left invariant vector fields. which will play the same role as constant coefficients vector fields in the Euclidean case.

Definition 1.1.7. *A vector field $\mathcal{X}$ on $\mathbb{H}^d$ is* left invariant *if it commutes with any left translation $\tau_w(w') \stackrel{\text{def}}{=} w \cdot w'$ which means*

$$\forall w \in \mathbb{H}^d\,,\ \forall f \in C^1(\mathbb{H}^d)\,,\ (\mathcal{X} \cdot f) \circ \tau_w = \mathcal{X} \cdot (f \circ \tau_w).$$

Proposition 1.1.4. *The set of left invariant vector fields on $\mathbb{H}^d$ is the $2d+1$ vectorial space generated by*

$$\mathcal{X}_j \stackrel{\text{def}}{=} \partial_{x_j} + 2\xi_j \partial_s\,,\ \Xi_j \stackrel{\text{def}}{=} \partial_{\xi_j} - 2x_j \partial_s \quad \text{and} \quad S \stackrel{\text{def}}{=} \partial_s \quad \text{for } j \text{ in } \{1, \cdots, d\}.$$

Proof. Let us fix some $C^1(\mathbb{R}^{2d+1})$ (which means for the classical notion on C^1 functions) real valued function f on $\mathbb{H}^d$. Written in terms of the differential of f, Definition 1.1.7 recasts in

$$\forall w \in \mathbb{H}^d\,,\ (Df \cdot \mathcal{X}) \circ \tau_w = D(f \circ \tau_w) \cdot \mathcal{X}.$$

Because the map τ_w is linear, the chain rule implies that

$$\forall (w, w') \in \mathbb{H}^d \times \mathbb{H}^d\,,\ Df(w \cdot w') \cdot \mathcal{X}(w \cdot w') = Df(w \cdot w') \circ D\tau_w \cdot \mathcal{X}(w').$$

As this identity must be satisfied for any function f, this gives in particular, choosing $w' = 0$, that

$$\forall w \in \mathbb{H}^d\,,\ \mathcal{X}(w) = D\tau_w\, \mathcal{X}(0). \tag{1.11}$$

By definition of τ_w,

$$D\tau_w(\dot{x}, \dot{\xi}, \dot{s}) = \big(\dot{x}, \dot{\xi}, \dot{s} + 2\langle \xi, \dot{x}\rangle - 2\langle \dot{\xi}, x\rangle\big),$$

which implies that

$$D\tau_w \cdot \partial_s = S\,,\ D\tau_w \cdot \partial_{x_j} = \mathcal{X}_j \quad \text{and} \quad D\tau_w \cdot \partial_{y_j} = \Xi_j.$$

The vector $\mathcal{X}(0)$ writes

$$\mathcal{X}(0) = \alpha_0 \partial_s + \sum_{j=1}^{d} \alpha_j \partial_{x_j} + \beta_j \partial_{\xi_j}.$$

Then using (1.11) gives the expected formula for $\mathcal{X}(w)$. Proving that, conversely, any linear combination of the vector fields S, $\mathcal{X}_j$ and Ξ_j is left invariant, is left to the reader. □

Notation. In all that follows, we denote $\mathcal{P} = (\mathcal{P}_1, \cdots, \mathcal{P}_{2d})$ with $\mathcal{P}_j \stackrel{\text{def}}{=} \mathcal{X}_j$ and $\mathcal{P}_{j+d} \stackrel{\text{def}}{=} \Xi_j$ for j in $\{1, \ldots, d\}$, we set for any multi-index α in $\{1, \ldots, 2d\}^k$:

$$\mathcal{P}^\alpha \stackrel{\text{def}}{=} \mathcal{P}_{\alpha_1} \ldots \mathcal{P}_{\alpha_k}. \tag{1.12}$$

Let us study the relation between let invariant derivatives and convolution.

Proposition 1.1.5. *If P is a left invariant vector field on $\mathbb{H}^d$, then we have for all smooth functions f and g with sufficient decay at infinity:*

$$P(f \star g) = f \star (P(g)).$$

Moreover, if g is even, that is $g(w^{-1}) = g(w)$ for all w in $\mathbb{H}^d$, then

$$P(f \star g) = (P(f)) \star g.$$

Proof. Thanks to the classical differentiation theorem, we have

$$P(f\star g)(w)=\int_{\mathbb{H}^d} f(v)P(g(v^{-1}\cdot w))dv.$$

As P is left invariant, we have

$$P(g(v^{-1}\cdot w))=(Pg)(v^{-1}\cdot w),$$

which yields the first relation.

In general $f\star(P(g))$ need not be equal to $(P(f))\star g$. Nevertheless, in the case when g is even, we have

$$\begin{aligned}(P(f)\star g)(w) &= \int_{\mathbb{H}^d}(Pf)(v)g(v^{-1}\cdot w)dv\\ &= \int_{\mathbb{H}^d}(Pf)(v)g(w^{-1}\cdot v)dv.\end{aligned}$$

An integration by parts and the fact that P is left invariant and divergence free leads to

$$(P(f)\star g)(w)=-\int_{\mathbb{H}^d} f(v)(Pg)(w^{-1}\cdot v)dv.$$

As g is even, Pg is odd. Thus we have

$$-(Pg)(w^{-1}\cdot v)=(Pg)(v^{-1}\cdot w)$$

and the proposition is proved. □

For a function f, the notation $\nabla_{\mathbb{H}}f$ designates $(\mathcal{P}_1 f,\cdots,\mathcal{P}_{2d}f)$.

Let us define the order of a differential operator (with respect to dilations).

Definition 1.1.8. *A left invariant differential operator D is said to be order k if for any C^1 function on $\mathbb{H}^d$, we have*

$$\forall a>0\,,\ D(f\circ\delta_a)=a^k(Df)\circ\delta_a.$$

According to this definition, the operators $\mathcal{X}_j$ and Ξ_j are first order, and the operator ∂_s is second order. Let us point out that this notion of order is different from the usual one in $\mathbb{R}^d$. To some extent, it may be compared with the case of the heat operator on $\mathbb{R}^{1+d}$, where ∂_t is 'equivalent' to two space derivatives, and is thus is of order 2.

A very important fact is that we have

$$[\mathcal{X}_j,\mathcal{X}_k]=[\Xi_j,\Xi_k]=0\quad\text{and}\quad[\Xi_k,\mathcal{X}_j]=4\delta_{j,k}S, \tag{1.13}$$

where $[A,B]\stackrel{\text{def}}{=}AB-BA$ denotes the *commutator* of the operators A and B. Let us emphasize that the last relation in (1.13) provides us with an example of a commutator of two differential operators of order 1, which is of order 2. In other words, in the Heisenberg group framework, we need not gain an order of differentiation by commutation. This will cause some difficulties in what follows.

1.2 The Laplace operator and the Sobolev spaces

The *Laplacian* associated to the vector fields $\mathcal{X}_j$ and Ξ_j, namely

$$\Delta_{\mathbb{H}} \stackrel{\text{def}}{=} \sum_{j=1}^{d} (\mathcal{X}_j^2 + \Xi_j^2) \tag{1.14}$$

plays a fundamental role in the Heisenberg group. It is the sum of the square of the elements of the canonical basis of left invariant differential operators of order 1. In terms of the usual derivatives, this operator writes

$$\Delta_{\mathbb{H}} f(x,\xi,s) = \Delta_{T^\star\mathbb{R}^d} f(x,\xi,s) + 2\sum_{j=1}^{d}(\xi_j\partial_{x_j} - x_j\partial_{\xi_j})\partial_s f(x,\xi,s) + 4|X|^2\partial_s^2 f(X,s). \tag{1.15}$$

One can now define Sobolev spaces with integer exponents as follows:

Definition 1.2.1. *For any nonnegative integer k, we denote by $H^k(\mathbb{H}^d)$ the subset of functions u in $L^2(\mathbb{H}^d)$ such that for all j in $\{0,\cdots,k\}$ and α in $\{1,\cdots,2d\}^j$, the function $\mathcal{P}^\alpha u$ belongs to $L^2(\mathbb{H}^d)$.*

Proposition 1.2.1. *The space $H^k(\mathbb{H}^d)$ endowed with the inner product*

$$(u|v)_{H^k(\mathbb{H}^d)} = \sum_{j=0}^{k}\sum_{\alpha\in\{1,\cdots,2d\}^j} (\mathcal{X}^\alpha u|\mathcal{X}^\alpha v)_{L^2}$$

is a Hilbert space, and the space $\mathcal{D}(\mathbb{H}^d)$ of test functions on $\mathbb{H}^d$ (that is smooth and compactly supported functions on $\mathbb{H}^d$) is dense in $H^k(\mathbb{H}^d)$.

Proof. In order to prove that the space $H^k(\mathbb{H}^d)$ is complete, let us consider a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ of $H^k(\mathbb{H}^d)$. Then $(\mathcal{X}^\alpha u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $L^2(\mathbb{H}^d)$ for any α in $\{1,\cdots,2d\}^j$ and any j in $\{0,\cdots,k\}$, and thus converges to some function u^α of $L^2(\mathbb{H}^d)$. Now, for all test function φ in $\mathcal{D}(\mathbb{H}^d)$, one may write that

$$\langle \mathcal{X}^\alpha u_n, \varphi\rangle = (-1)^{|\alpha|}\langle u_n, \mathcal{X}^\alpha\varphi\rangle,$$

whence, denoting by u the limit of $(u_n)_{n\in\mathbb{N}}$ in $L^2(\mathbb{H}^d)$,

$$\lim_{n\to+\infty}\langle \mathcal{X}^\alpha u_n, \varphi\rangle = (-1)^{|\alpha|}\langle u, \mathcal{X}^\alpha\varphi\rangle = \langle \mathcal{X}^\alpha u, \varphi\rangle$$

where $\langle\cdot,\cdot\rangle$ denotes the classical duality bracket between distribution and test functions. This means that $u^\alpha = \mathcal{X}^\alpha u$. Consequently, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $H^k(\mathbb{H}^d)$.

In order to prove the density of $\mathcal{D}(\mathbb{H}^d)$ in $H^k(\mathbb{H}^d)$, we mimic the proof of the corresponding result for Sobolev spaces on $\mathbb{R}^n$. More concretely, we fix some bump function θ in $\mathcal{D}(\mathbb{R})$ with value 1 on $[-1,1]$ and set

$$u_n \stackrel{\text{def}}{=} \theta\big(\rho(\delta_{n^{-1}}\cdot)\big)\,\big(\chi_{n^{-1}}\star u\big) \quad \text{for} \quad n\geq 1,$$

where the approximation of identity $(\chi_\varepsilon)_{\varepsilon>0}$ has been defined in Lemma 1.1.1.

Next, we write $u_n - u = v_n + w_n$ with

$$v_n \stackrel{\text{def}}{=} \big(\theta\big(\rho(\delta_{n^{-1}}\cdot)\big) - 1\big)u \quad \text{and} \quad w_n \stackrel{\text{def}}{=} \theta\big(\rho(\delta_{n^{-1}}\cdot)\big)\,\big(\chi_{n^{-1}} \star u - u\big).$$

Leibniz' rule implies that

$$\mathcal{X}^\alpha v_n = \big(\theta\big(\rho(\delta_{n^{-1}}\cdot)\big) - 1\big)\mathcal{X}^\alpha u + \sum_{\substack{\beta+\gamma=\alpha\\ \beta\neq 0}} \frac{\alpha!}{\beta!\gamma!}\mathcal{X}^\beta\big(\theta\big(\rho(\delta_{n^{-1}}\cdot)\big)\,\mathcal{X}^\gamma u.$$

Lebesgue dominated convergence theorem implies that the first term tends to 0 in $L^2(\mathbb{H}^d)$, and it is clear that the sum is $\mathcal{O}(n^{-1})$ in $L^2(\mathbb{H}^d)$. Therefore $v_n \to 0$ in $L^2(\mathbb{H}^d)$. Similarly, for w^n we have,

$$\mathcal{X}^\alpha w_n = \theta\big(\rho(\delta_{n^{-1}}\cdot)\big)(\chi_{n^{-1}} \star \mathcal{X}^\alpha u - \mathcal{X}^\alpha u) + \sum_{\substack{\beta+\gamma=\alpha\\ \beta\neq 0}} \frac{\alpha!}{\beta!\gamma!}\mathcal{X}^\beta\big(\theta\big(\rho(\delta_{n^{-1}}\cdot)\big)\,\big(\chi_{n^{-1}} \star \mathcal{X}^\gamma u - \mathcal{X}^\gamma u).$$

According to Lemma 1.1.1, the first term tends to 0 in L^2, and, as for v_n, the sum is $\mathcal{O}(n^{-1})$ in $L^2(\mathbb{H}^d)$. As u_n is obviously in $\mathcal{D}(\mathbb{H}^d)$, this completes the proof of the density result. □

Theorem 1.2.1. *Operator $\Delta_{\mathbb{H}}$ with domain $H^2(\mathbb{H}^d)$ is self-adjoint on $L^2(\mathbb{H}^d)$, and for any u in $H^2(\mathbb{H}^d)$, we have*[1]

$$\|u\|^2_{H^2(\mathbb{H}^d)} \sim \|u\|^2_{L^2(\mathbb{H}^d)} + \|\Delta_{\mathbb{H}} u\|^2_{L^2(\mathbb{H}^d)}.$$

Proof. As $\Delta_{\mathbb{H}}$ is symmetric with domain $H^2(\mathbb{H}^d)$, we just have to prove that the domain of the adjoint operator $(\Delta_{\mathbb{H}})^\star$ is $H^2(\mathbb{H}^d)$, that is too say, for any $u \in L^2(\mathbb{H}^d)$,

$$\Big(\forall v \in H^2(\mathbb{H}^d)\,,\ (u|\Delta_{\mathbb{H}} v)_{L^2} \leq C\|v\|_{L^2}\Big) \Longrightarrow u \in H^2(\mathbb{H}^d). \tag{1.16}$$

By an omitted density argument (use Proposition 1.2.1), it amounts to proving that

$$\forall u \in H^2(\mathbb{H}^d)\,,\ \|u\|^2_{H^2(\mathbb{H}^d)} \leq C\big(\|u\|^2_{L^2} + \|\Delta_{\mathbb{H}} u\|^2_{L^2}\big). \tag{1.17}$$

By integration by parts, we have immediately that

$$\|\nabla_{\mathbb{H}^d} u\|^2_{L^2} = -(u|\Delta_{\mathbb{H}} u)_{L^2} \leq \|u\|_{L^2}\|\Delta_{\mathbb{H}} u\|_{L^2}. \tag{1.18}$$

Now we have to control the second order derivatives. This is based on the following lemma which is is the cornerstone of the theory of subelliptic operators.

Lemma 1.2.1. *For α in $\mathbb{R}$, let us define the operator Λ^α acting on smooth functions of $\mathbb{H}^d$ by*

$$\Lambda^\alpha a \stackrel{\text{def}}{=} \mathcal{F}^{-1}_{\mathbb{R}^{2d+1}}\big(|\sigma|^\alpha \mathcal{F}_{\mathbb{R}^{2d+1}} a(\xi,\eta,\sigma)\big)$$

where $\mathcal{F}_{\mathbb{R}^{2d+1}}$ stands for the standard Fourier transform on $\mathbb{R}^{2d+1}$. Then we have for any u in $\mathcal{D}(\mathbb{H}^d)$,

$$\|\Lambda^{\frac12} u\|^2_{L^2} \leq \frac{1}{4d}(-\Delta_{\mathbb{H}} u|u)_{L^2} = \frac{1}{4d}\|\nabla_{\mathbb{H}} u\|^2_{L^2}. \tag{1.19}$$

Moreover, we have

$$\|Su\|^2_{L^2} \leq \frac{1}{4d}\|\Delta_{\mathbb{H}} u\|^2_{L^2}. \tag{1.20}$$

[1]From now on, we agree that the notation $a \sim b$ means that $C^{-1}a \leq b \leq Ca$ for some harmless positive constant C.

Proof. As obviously $\|\Lambda^{\frac12}u\|_{L^2}=\|\Lambda^{-\frac12}Su\|_{L^2}$, we get, using the commutation relation (1.13),

$$\Lambda^{-\frac12}Su=\frac14\Lambda^{-\frac12}[\Xi_j,\mathcal{X}_j]u\quad\text{and thus}\quad\|\Lambda^{\frac12}u\|_{L^2}^2=\frac{1}{16d}\sum_{j=1}^{d}\big\|\Lambda^{-\frac12}[\Xi_j,\mathcal{X}_j]u\big\|_{L^2}^2.$$

As Λ^α is symmetric on L^2 and commutes with $\mathcal{X}_j$ and Ξ_j, and as S commutes with $\mathcal{X}_j$ and Ξ_j, we get by integration by parts,

$$\begin{aligned}\|\Lambda^{-\frac12}[\Xi_j,\mathcal{X}_j]u\|_{L^2}^2&=\big([\Xi_j,\mathcal{X}_j]u|\Lambda^{-1}[\Xi_j,\mathcal{X}_j]u\big)_{L^2}\\&=-\big(\mathcal{X}_ju|\Lambda^{-1}[\Xi_j,\mathcal{X}_j]\Xi_ju\big)_{L^2}+\big(\Xi_ju|\Lambda^{-1}[\Xi_j,\mathcal{X}_j]\mathcal{X}_ju\big)_{L^2}.\end{aligned}$$

Using that $\Lambda^{-1}[\Xi_j,\mathcal{X}_j]$ is a bounded operator on L^2 the norm of which is less than 4, (it is indeed the Fourier multiplier $4i\mathrm{sg}\lambda$ in the Fourier space associated to s) we deduce that

$$\begin{aligned}\|\Lambda^{-\frac12}[\Xi_j,\mathcal{X}_j]u\|_{L^2}^2&\leq 8\|\mathcal{X}_ju\|_{L^2}\|\Xi_ju\|_{L^2}\\&\leq 4(\|\mathcal{X}_ju\|_{L^2}^2+\|\Xi_ju\|_{L^2}^2).\end{aligned}$$

Thus, because $\mathcal{X}_j$ and Ξ_j are divergence free, we have, by integration by parts

$$\begin{aligned}\|\Lambda^{\frac12}u\|_{L^2}^2&\leq\frac{1}{4d}\sum_{j=1}^{2d}(-\mathcal{X}_j^2u|u)_{L^2}\\&\leq\frac{1}{4d}(-\Delta_{\mathbb{H}}u|u)_{L^2},\end{aligned}$$

which proves the first part of the lemma. Now, applying the above inequality with $u=\Lambda^{\frac12}u$. Using that $\Lambda^{\frac12}$ and $\Delta_{\mathbb{H}}$ commute, we can write

$$\begin{aligned}\|Su\|_{L^2}^2&=\|\Lambda u\|_{L^2}^2\\&\leq-\frac{1}{4d}(\Delta_{\mathbb{H}}\Lambda^{\frac12}u|\Lambda^{\frac12}u)_{L^2}\\&\leq-\frac{1}{4d}(\Delta_{\mathbb{H}}u|\Lambda u)_{L^2}.\end{aligned}$$

Using Cauchy-Schwartz inequality, we get

$$\|Su\|_{L^2}^2\leq\frac{1}{4d}\|\Delta_{\mathbb{H}}u\|_{L^2}\|\Lambda u\|_{L^2}.$$

As $\|Su\|_{L^2}^2=\|\Lambda u\|_{L^2}^2$ this proves the second estimate of the lemma. □

Continuation of the proof of Theorem 1.2.1. Completing the proof is essentially algebraic matter. This is contained in the following lemma.

Lemma 1.2.2. *Let us consider two divergence free vector fields $\mathcal{Z}_1$ and $\mathcal{Z}_2$ (with real value coefficients) which both commute with $[\mathcal{Z}_1,\mathcal{Z}_2]$. Then we have*

$$\|\mathcal{Z}_1\mathcal{Z}_2u\|_{L^2}^2=\|[\mathcal{Z}_1,\mathcal{Z}_]u\|_{L^2}^2+(\mathcal{Z}_1^2u|\mathcal{Z}_2^2u)_{L^2}.\tag{1.21}$$

Proof. Let us write that

$$\begin{aligned}\|\mathcal{Z}_1\mathcal{Z}_2u\|_{L^2}^2&=(\mathcal{Z}_1\mathcal{Z}_2u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2}\\&=([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2}+(\mathcal{Z}_2\mathcal{Z}_1u|\mathcal{Z}_1\mathcal{Z}_2u)_{L^2}.\end{aligned}$$

From integration by parts with respect to $\mathcal{Z}_2$ and to $\mathcal{Z}_1$, we infer that

$$
\begin{aligned}
\|\mathcal{Z}_1\mathcal{Z}_2 u\|_{L^2}^2 &= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2 u)_{L^2} - (\mathcal{Z}_1 u|\mathcal{Z}_2\mathcal{Z}_1\mathcal{Z}_2 u)_{L^2}\\
&= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2 u)_{L^2} - (\mathcal{Z}_1 u|[\mathcal{Z}_2,\mathcal{Z}_1]\mathcal{Z}_2 u)_{L^2} - (\mathcal{Z}_1 u|\mathcal{Z}_1\mathcal{Z}_2^2 u)_{L^2}\\
&= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2 u)_{L^2} - (\mathcal{Z}_1 u|[\mathcal{Z}_2,\mathcal{Z}_1]\mathcal{Z}_2 u)_{L^2} + (\mathcal{Z}_1^2 u|\mathcal{Z}_2^2 u)_{L^2}.
\end{aligned}
$$

As $\mathcal{Z}_2$ commutes with $[\mathcal{Z}_1,\mathcal{Z}_2]$, we have by integrations by parts with respect to $\mathcal{Z}_1$ and $\mathcal{Z}_2$,

$$
\begin{aligned}
([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2 u)_{L^2} - (\mathcal{Z}_1 u|[\mathcal{Z}_2,\mathcal{Z}_1]\mathcal{Z}_2 u)_{L^2} &= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2 u)_{L^2} + (\mathcal{Z}_2\mathcal{Z}_1 u|[\mathcal{Z}_2,\mathcal{Z}_1]u)_{L^2}\\
&= ([\mathcal{Z}_1,\mathcal{Z}_2]u|\mathcal{Z}_1\mathcal{Z}_2 u)_{L^2} - (\mathcal{Z}_1\mathcal{Z}_2 u|[\mathcal{Z}_2,\mathcal{Z}_1]u)_{L^2}\\
&\quad + \|[\mathcal{Z}_2,\mathcal{Z}_1]u\|_{L^2}^2.
\end{aligned}
$$

This concludes the proof of the lemma. □

Conclusion of the proof of Theorem 1.2.1. Let us apply this identity to the vector fields $\mathcal{X}_j$ or Ξ_k involved in the definition of $\|u\|^2_{\dot H^2(\mathbb{H}^d)}$. In light of the commutation properties pointed out in (1.13), we get

$$
\begin{aligned}
\|u\|^2_{\dot H^2(\mathbb{H}^d)} &= \sum_{1\leq j,k\leq d}\big(\|\mathcal{X}_j\mathcal{X}_k u\|_{L^2}^2 + \|\Xi_j\mathcal{X}_k u\|_{L^2}^2 + \|\mathcal{X}_j\Xi_k u\|_{L^2}^2 + \|\Xi_j\Xi_k u\|_{L^2}^2\big)\\
&\leq 2d(d-1)\|Su\|_{L^2}^2 + \sum_{1\leq j,k\leq d}\big((\mathcal{X}_j^2 u|\mathcal{X}_k^2 u)_{L^2} + (\Xi_j^2 u|\mathcal{X}_k^2 u)_{L^2}\\
&\qquad\qquad + (\mathcal{X}_j^2 u|\Xi_k^2 u)_{L^2} + (\Xi_j^2 u|\Xi_k^2 u)_{L^2}\big)\\
&\leq 2d(d-1)\|Su\|_{L^2}^2 + \|\Delta_{\mathbb{H}} u\|_{L^2}^2.
\end{aligned}
$$

Applying Inequality (1.20) implies the theorem. □

Lemma 1.2.1 implies immediately the following corollary

Corollary 1.2.1. *The space $H^1(\mathbb{H}^d)$ is continuously included in the space $L^2(\mathbb{R}^{2d}; H^{\frac12}(\mathbb{R}))$ and also in the space $H^{\frac12}_{\mathrm{loc}}(\mathbb{R}^{2d+1})$.*

Proof. The first embedding is simply the translation of Inequality (1.19). Let us write that

$$\partial_{x_j} u = \mathcal{X}_j u - 2\xi_j\partial_s u \quad\text{and}\quad \partial_{\xi_j} u = \Xi_j u + 2x_j\partial_s u.$$

Using again Inequality (1.19), this implies that the functions

$$-2\xi_j\partial_s u \quad\text{and}\quad 2x_j\partial_s u$$

belongs locally to $H^{-\frac12}(\mathbb{R}^{2d+1})$. As the functions $\mathcal{X}_j u$ and $\Xi_j u$ are in $L^2(\mathbb{R}^{2d+1})$, the corollary is proved. □

It is possible to describe the spectrum of the self adjoint operator $-\Delta_{\mathbb{H}}$. This is the aim of the following proposition.

Theorem 1.2.2. *The spectrum of the self adjoint operator $-\Delta_{\mathbb{H}}$ is the interval $[0,+\infty[$.*

Before proving the result, let us recall that in the classical Euclidean case, the Fourier transform allows to prove very easily this result. Indeed, for any given real number α_0, let us consider a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{S}(\mathbb{R}^n)$ such that

$$\|f_n\|_{L^2} = 1 \quad\text{and}\quad \operatorname{Supp}\widehat{f}_n \subset \Big\{\xi\in\mathbb{R}^n \,/\, \big||\xi| - |\alpha_0|\big| \leq \frac1n\Big\}.$$

As we have $-\Delta e^{i\langle\xi,x\rangle} = |\xi|^2 e^{i\langle\xi,x\rangle}$, we get that

$$
\begin{aligned}
(2\pi)^d \| -\Delta f_n - \alpha_0^2 f_n \|_{L^2}^2 &= \int_{\mathbb{R}^n} (|\xi|^2 - |\alpha_0|^2)^2 |\widehat{f}_n(\xi)|^2 d\xi \\
&\leq \frac{1}{n^2}\left(2|\alpha_0| + \frac{1}{n}\right)^2.
\end{aligned}
$$

This proves that $-\Delta - \alpha_0^2 \mathrm{Id}_{\mathbb{R}^n}$ cannot have a continuous inverse and that the spectrum of $-\Delta$ thus contains the interval $[0,+\infty[$ (the converse being obvious). In fact, the Fourier transform provides us with a description of the spectral measure.

Proof of Theorem 1.2.2. Note that (1.18) implies that the spectrum of $-\Delta_{\mathbb{H}}$ is included in $[0,+\infty[$. To prove the other inclusion, introduce the following function

$$
\Theta_\lambda : \left\{ \begin{array}{ccc} \mathbb{H}^d & \to & \mathbb{C} \\ (Y,s) & \longmapsto & e^{is\lambda} e^{-\lambda|Y|^2}, \end{array} \right.
$$

which will play an analogous role here as the function $x \mapsto e^{-i\langle\xi,x\rangle}$ in the $\mathbb{R}^n$ case. We claim that

$$
-\Delta_{\mathbb{H}} \Theta_\lambda = 4\lambda d \Theta_\lambda \tag{1.22}
$$

which is the analog of $-\Delta_x e^{i\langle\xi,x\rangle} = |\xi|^2 e^{i\langle\xi,x\rangle}$. Let us check this formula. We have

$$
\begin{aligned}
&\mathcal{X}_j \Theta_\lambda = -2\lambda(y_j - i\eta_j)\Theta_\lambda \quad \text{and thus} \quad \mathcal{X}_j^2 \Theta_\lambda = \big(-2\lambda + 4\lambda^2 (y_j - i\eta_j)^2\big)\Theta_\lambda \quad \text{and} \\
&\Xi_j \Theta_\lambda = -2\lambda(\eta_j + i y_j)\Theta_\lambda \quad \text{and thus} \quad \Xi_j^2 \Theta_\lambda = \big(-2\lambda + 4\lambda^2 (\eta_j + i y_j)^2\big)\Theta_\lambda,
\end{aligned}
$$

which obviously gives Formula (1.22). Then for a given function χ of $\mathcal{D}(]0,\infty[)$, let us define the function $T\chi$ from $\mathbb{H}^d$ to $\mathbb{C}$ by

$$
T\chi(Y,s) = \sqrt{2\pi}\, \pi^{-\frac{d}{2}} \mathcal{F}_R^{-1}\big(\chi e^{-\cdot|Y|^2}\big).
$$

Using Fourier Plancherel theorem for the Fourier transform on $\mathbb{R}$, we get

$$
\begin{aligned}
\|T\chi\|_{L^2}^2 &= \pi^{-d} \int_{T^\star \mathbb{R}^d \times \mathbb{R}} \chi^2(\lambda) e^{-2\lambda|Y|^2} \lambda^d \, dY \, d\lambda \\
&= \|\chi\|_{L^2(\mathbb{R})}^2.
\end{aligned}
$$

Moreover, Relation (1.22) implies that

$$
-\Delta_{\mathbb{H}} T\chi = T\widetilde{\chi} \quad \text{with} \quad \widetilde{\chi}(\lambda) \stackrel{\text{def}}{=} 4\lambda d \chi(\lambda). \tag{1.23}
$$

Now for any non negative real number λ_0, let us consider a sequence $(\chi_n)_{n\in\mathbb{N}}$ of functions of $\mathcal{D}(]0,\infty[)$ such that

$$
\int_{\mathbb{R}} \chi_n^2(\lambda)\, d\lambda = 1 \quad \text{and} \quad \mathrm{Supp}\ \chi_n \subset \left]\lambda_0, \lambda_0 + \frac{1}{n}\right[.
$$

Then we have that $\|T_{\chi_n}\|_{L^2}^2 = 1$ and, using Equality (1.23),

$$
\begin{aligned}
\| -\Delta_{\mathbb{H}} T_{\chi_n} - 4\lambda_0 d\, T_{\chi_n} \|_{L^2(\mathbb{H}^d)}^2 &= \|T_{\widetilde{\chi}_n - 4\lambda_0 \chi_n}\|_{L^2(\mathbb{R})}^2 \\
&= \int_{\mathbb{R}} 16(\lambda - \lambda_0)^2 d^2 \chi_n^2(\lambda)\, d\lambda \\
&\leq \frac{16d^2}{n^2},
\end{aligned}
$$

and thus $-\Delta_{\mathbb{H}} - 4\lambda_0 d\, \mathrm{Id}_{\mathbb{H}^d}$ cannot have a continuous inverse. As the spectrum of a self-adjoint operator is closed subset of $\mathbb{R}$, the theorem is proved. $\square$

Let us conclude this section by a theorem which is a generalization of Theorem 1.2.1 and wich will be useful in the nex section.

Theorem 1.2.3. *For any positive integer ℓ, we have*

$$\|\Delta_{\mathbb{H}}^{\ell} u\|_{L^2}^2 \leq \|u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2 \leq C_\ell^2 \|\Delta_{\mathbb{H}}^{\ell} u\|_{L^2}^2,$$

with $C_\ell = (Cd)^\ell \, \ell!$ for some absolute constant C.

Proof. The left inequality being obvious, let us focus on the proof of the right inequality. We proceed by induction on ℓ, the case $\ell = 0$ being trivial. So let us now assume that the theorem holds for some nonnegative integer ℓ. By definition of the Sobolev semi-norms, we have, using the induction hypothesis,

$$\begin{aligned}\|u\|_{\dot{H}^{2\ell+2}(\mathbb{H}^d)}^2 &= \sum_{1\leq j,k\leq d} \big(\|\mathcal{X}_j\mathcal{X}_k u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2 + \|\Xi_j\mathcal{X}_k u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2 \\ &\qquad\qquad + \|\mathcal{X}_j\Xi_k u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2 + \|\Xi_j\Xi_k u\|_{\dot{H}^{2\ell}(\mathbb{H}^d)}^2\big) \\ &\leq C_\ell \sum_{1\leq j,k\leq d} \big(\|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}_j\mathcal{X}_k u\|_{L^2}^2 + \|\Delta_{\mathbb{H}}^{\ell}\Xi_j\mathcal{X}_k u\|_{L^2}^2 + \|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}_j\Xi_k u\|_{L^2}^2 + \|\Delta_{\mathbb{H}}^{\ell}\Xi_j\Xi_k u\|_{L^2}^2\big)\cdot \end{aligned} \tag{1.24}$$

Now, we have to prove a generalization of Identity (1.21). Let us consider $\mathcal{X}$ and $\mathcal{Y}$ two divergence free vector fields such that $\mathcal{X}$, $\mathcal{Y}$ and $\Delta_{\mathbb{H}}$ commute with $[\mathcal{X}, \mathcal{Y}]$. Omitted computations lead to

$$\|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}\mathcal{Y}u\|_{L^2}^2 = (\Delta_{\mathbb{H}}^{\ell}\mathcal{X}^2 u|\Delta_{\mathbb{H}}^{\ell}\mathcal{Y}^2 u)_{L^2} + \sum_{m=1}^{6} R_m(u), \tag{1.25}$$

where the terms $R_m(u)$ are defined by

$$\begin{aligned} R_1(u) &\stackrel{\text{def}}{=} \big(\Delta_{\mathbb{H}}^{\ell}[\mathcal{X},\mathcal{Y}]u|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}\mathcal{Y}u\big)_{L^2} \\ R_2(u) &\stackrel{\text{def}}{=} \big([\Delta_{\mathbb{H}}^{\ell},\mathcal{Y}]\mathcal{X}u|\Delta_{\mathbb{H}}^{\ell}\mathcal{X}\mathcal{Y}u\big)_{L^2} \\ R_3(u) &\stackrel{\text{def}}{=} \big(\mathcal{Y}\Delta_{\mathbb{H}}^{\ell}\mathcal{X}u|[\Delta_{\mathbb{H}}^{\ell},\mathcal{X}]\mathcal{Y}u\big)_{L^2} \\ R_4(u) &\stackrel{\text{def}}{=} \big(\Delta_{\mathbb{H}}^{\ell}\mathcal{X}u|\Delta_{\mathbb{H}}^{\ell}[\mathcal{X},\mathcal{Y}]\mathcal{Y}u\big)_{L^2} \\ R_5(u) &\stackrel{\text{def}}{=} \big([\mathcal{X},\Delta_{\mathbb{H}}^{\ell}]\mathcal{X}u|\mathcal{Y}\Delta_{\mathbb{H}}^{\ell}\mathcal{Y}u\big)_{L^2} \\ R_6(u) &\stackrel{\text{def}}{=} \big(\Delta_{\mathbb{H}}^{\ell}\mathcal{X}^2 u|[\mathcal{Y},\Delta_{\mathbb{H}}^{\ell}]\mathcal{Y}u\big)_{L^2}. \end{aligned}$$

Taking advantage the fact that

$$[\mathcal{X}_j, \Delta_{\mathbb{H}}] = -8\Xi_j S \quad\text{and}\quad [\Xi_j, \Delta_{\mathbb{H}}] = 8\mathcal{X}_j S. \tag{1.26}$$

and of the fact that $\mathcal{X}_j$ and Ξ_j commute with S, and using that

$$[\mathcal{X}_j, \Delta_{\mathbb{H}}^{\ell}] = \sum_{m=0}^{\ell-1} \Delta_{\mathbb{H}}^{m}[\mathcal{X}_j, \Delta_{\mathbb{H}}]\Delta_{\mathbb{H}}^{\ell-m-1} \quad\text{for}\quad j = 1, \cdots, d,$$

we infer that

$$[\mathcal{X}_j, \Delta_{\mathbb{H}}^{\ell}] = -8\Big(\sum_{m=0}^{\ell-1} \Delta_{\mathbb{H}}^{m}\Xi_j\Delta_{\mathbb{H}}^{\ell-m-1}\Big)S \quad\text{and}\quad [\Xi_j, \Delta_{\mathbb{H}}^{\ell}] = 8\Big(\sum_{m=0}^{\ell-1} \Delta_{\mathbb{H}}^{m}\mathcal{X}_j\Delta_{\mathbb{H}}^{\ell-m-1}\Big)S. \tag{1.27}$$

Thus, if $\mathcal{X}$ and $\mathcal{Y}$ belong to the family $(\mathcal{X}_j)_{1\leq j\leq 2d}$ then all the terms $R_m(u)$ may be written (using (1.13), (1.27), and performing integration by parts as the case may be) as a linear combination of terms of the type

$$(\mathcal{X}^\alpha u|\mathcal{X}^\beta Su)_{L^2} \quad \text{with} \quad (\alpha,\beta)\in \mathcal{I}^{2\ell+2}\times\mathcal{I}^{2\ell}.$$

Note that there are at most $\mathcal{O}((\ell+1)^2)$ such terms. Now, by definition of Sobolev norms, we have for (α,β) in $\mathcal{I}^{2\ell+2}\times\mathcal{I}^{2\ell}$,

$$|(\mathcal{X}^\alpha u|\mathcal{X}^\beta Su)_{L^2}|\leq \|u\|_{\dot H^{2\ell+2}(\mathbb{H}^d)}\|Su\|_{\dot H^{2\ell}(\mathbb{H}^d)}.$$

Using Inequality (1.24) and Identity (1.25), we deduce that for some universal constant C,

$$\|u\|^2_{\dot H^{2\ell+2}(\mathbb{H}^d)}\leq C_\ell\big(\|\Delta_{\mathbb{H}}^{\ell+1}u\|^2_{L^2}+Cd^2(\ell+1)^2\|u\|_{\dot H^{2\ell+2}(\mathbb{H}^d)}\|Su\|_{\dot H^{2\ell}(\mathbb{H}^d)}\big).$$

The induction hypothesis implies that

$$\|u\|^2_{\dot H^{2\ell+2}(\mathbb{H}^d)}\leq C_\ell\big(\|\Delta_{\mathbb{H}}^{\ell+1}u\|^2_{L^2}+CC_\ell d^2(\ell+1)^2\|u\|_{\dot H^{2\ell+2}(\mathbb{H}^d)}\|\Delta_{\mathbb{H}}^{\ell}Su\|_{L^2}\big).$$

As S and $\Delta_{\mathbb{H}}$ commute, we get, using Inequality (1.20)

$$\|u\|^2_{\dot H^{2\ell+2}(\mathbb{H}^d)}\leq C^2C_\ell^2d^2(\ell+1)^2\|\Delta_{\mathbb{H}}^{\ell+1}u\|^2_{L^2}$$

and the theorem is proved. $\square$

1.3 Remarks on the Schwartz space on the Heisenberg group

We conclude this chapter with some remarks on the Schwartz space of smooth rapidly decreasing functions of $\mathbb{H}^d$. The following proposition states that the usual semi-norms on the Schwartz class and the semi-norms using the structure of $\mathbb{H}^d$ are equivalent.

Proposition 1.3.1. *Let us introduce the notations*

$$(M_{\mathbb{H}}f)(X,s)\stackrel{\text{def}}{=}(|X|^2-is)f(X,s).$$

Moreover if α is in $\mathbb{N}^{1+2d}$, we define

$$w^\alpha\stackrel{\text{def}}{=}s^{\alpha_0}x_1^{\alpha_1}\cdots x_d^{\alpha_d}\xi_1^{\alpha_{d+1}}\cdots\xi_d^{\alpha_{2d}} \quad \textit{and} \quad \widetilde{|\alpha|}\stackrel{\text{def}}{=}2\alpha_0+\alpha_1\cdots+\alpha_{2d}.$$

Then the three families of semi-norms defined on $\mathcal{S}(\mathbb{H}^d)$ by

$$\begin{aligned}
\|f\|^2_{k,\mathcal{S}(\mathbb{H}^d)} &\stackrel{\text{def}}{=} \|f\|^2_{L^2}+\|M_{\mathbb{H}}^kf\|^2_{L^2}+\big\|\Delta_{\mathbb{H}}^kf\big\|^2_{L^2}\\
N_k^2(f) &\stackrel{\text{def}}{=} \sum_{\widetilde{|\alpha|}+|\beta|\leq k}\big\|w^\alpha\mathcal{X}^\beta f\big\|^2_{L^2} \quad \textit{and}\\
\widetilde{N}_k^2(f) &\stackrel{\text{def}}{=} \sum_{\widetilde{|\alpha|}+|\beta|\leq k}\big\|\mathcal{X}^\beta(w^\alpha f)\big\|^2_{L^2} \quad \textit{and}
\end{aligned}$$

are all equivalent to the classical semi norm on $\mathcal{S}(\mathbb{R}^{2d+1})$.

Proof. Observe that $\mathcal{X}^\gamma w^{\gamma'}$ is an homogeneous polynomial of degree $\gamma'-\gamma$, and equal to 0 if the length of γ is greater than the length of γ'. Thus, thanks to Leibniz' rule, we have

$$[\mathcal{X}^\beta, w^\alpha]f(w) = \sum_{\substack{\widetilde{|\alpha|}\leq\widetilde{|\alpha|}-1\\ |\beta'|\leq|\beta|-1}} a_{\alpha,\alpha',\beta',\beta}\, w^{\alpha'}\mathcal{X}^{\beta'}f(w). \tag{1.28}$$

Then an omitted induction implies that

$$C_k^{-1}\widetilde{N}_k(f)\leq N_k(f)\leq C_k\widetilde{N}_k(f). \tag{1.29}$$

Let us observe that Leibniz' rule implies that, if $\widetilde{|\alpha|}+|\beta|\leq k$, then

$$\mathcal{X}^\beta(w^\alpha f)(w) = w^\alpha\mathcal{X}^\beta f(w) + \sum_{\widetilde{|\alpha'|}+|\beta'|\leq k-2} a_{\alpha',\beta'}w^{\alpha'}\mathcal{X}^{\beta'}f(w). \tag{1.30}$$

As obviously $\|f\|_k$ is less than or equal to $N_{2k}(f)$, the proof is the fact that the three semi-norms are equivalent reduces to the proof of

$$\forall k\in\mathbb{N}\,,\ \exists(C_k,M_k)\ /\ \forall f\in\mathcal{S}(\mathbb{H}^d)\,,\ N_k(f)\leq C_k\|f\|^2_{k,\mathcal{S}(\mathbb{H}^d)}. \tag{1.31}$$

Using an integration by parts, we get

$$\int_{\mathbb{H}^d} w^\alpha\mathcal{X}^\beta f(w)w^\alpha\mathcal{X}^\beta\overline{f}(w)\,dw = (-1)^{|\beta|}\int_{\mathbb{H}^d} f(w)\mathcal{X}^\beta\big(w^{2\alpha}\mathcal{X}^\beta\overline{f}(w)\big)dw$$

Applying (1.28), we get that

$$\int_{\mathbb{H}^d} f(w)\mathcal{X}^\beta\big(w^{2\alpha}\overline{\mathcal{X}}^\beta f(w)\big)dw = \sum_{\substack{\widetilde{|\alpha'|}\leq 2|\alpha|\\ |\beta'|\leq|\beta|}} a_{\alpha,\alpha',\beta,\beta'}\int_{\mathbb{H}^d} w^{\alpha'}f(w)\mathcal{X}^{\beta'}\mathcal{X}^\beta\overline{f}(w)\,dw$$

Thanks to Cauchy-Schwartz inequality and by definition of $M_{\mathbb{H}}$, we get, applying Theorem 1.2.3,

$$\begin{aligned}\sum_{\substack{\widetilde{|\alpha'|}\leq 2|\alpha|\\ |\beta'|\leq|\beta|}} a_{\alpha,\alpha',\beta,\beta'}\int_{\mathbb{H}^d} w^{\alpha'}f(w)\mathcal{X}^{\beta'}\mathcal{X}^\beta\overline{f}(w)\,dw \ &\lesssim\ \|(\mathrm{Id}+M^k_{\mathbb{H}}f\|_{L^2}\|f\|_{H^{2k}(\mathbb{H}^d)}\\ &\lesssim\ \|f\|^2_{L^2}+\|M^k_{\mathbb{H}}f\|^2_{L^2}+\|\Delta^k_{\mathbb{H}}f\|^2_{L^2(\mathbb{H}^d)}.\end{aligned}$$

This proves that the three families of equivalent. In order to prove that there are equivalent to the classical family, let us observe that as

$$S=\frac{1}{4}[\Xi_j,\mathcal{X}_j]\,,\partial_{x_j}=\mathcal{X}_j-\frac{y_j}{2}\left(\Xi_j\mathcal{X}_j-\mathcal{X}_j\Xi_j\right)\quad\text{and}\quad\partial_{y_j}=\Xi_j+\frac{x_j}{2}\left(\Xi_j\mathcal{X}_j-\mathcal{X}_j\Xi_j\right)$$

and, for all j in $\{1,\cdots,d\}$, we infer that

$$\widetilde{\|f\|}_{k,\mathcal{S}(\mathbb{R}^{2d+1})}\leq C\|f\|_{2k,\mathcal{S}(\mathbb{H}^d)}$$

where $\widetilde{\|f\|}^2_{k,\mathcal{S}(\mathbb{R}^{2d+1})}\stackrel{\text{def}}{=}\displaystyle\sum_{|\alpha|+|\beta|\leq k}\|x^\alpha\partial_\beta f(x)\|^2_{L^2(\mathbb{R}^{2d+1})}$ which ends the proof of the proposition.

□

In all that follows, we shall use indistinctly the above three different types of semi-norms, and the Schwartz class on $\mathbb{R}^{2d+1}$ (or, equivalently, on $\mathbb{H}^d$) will be sometimes just denoted by $\mathcal{S}$. When we denote $\mathcal{S}(\mathbb{H}^d)$ we have in mind to use the semi-norm related to the structure of the Heisenberg group. The only property we have to prove here concerns the convolution in the sense of the Heisenberg group.

As a conclusion, let us prove a continuity theorem about the convolution which will be useful when we shall study the Fourier transform of distribution.

Proposition 1.3.2. *Let us define $L^1_{\mathcal{S}}(\mathbb{H}^d)$ the space of integrable functions f on $\mathbb{H}^d$ such that*

$$\forall k \in \mathbb{N}\,,\ \|f\|_{k,L^1_{\mathcal{S}}(\mathbb{H}^d)} \stackrel{\text{def}}{=} \|M^k_{\mathbb{H}} f\|_{L^1(\mathbb{H}^d)} < \infty.$$

The convolution product on $\mathbb{H}^d$ is bicontinuous operator from $L^1_{\mathcal{S}}(\mathbb{H}^d) \times \mathcal{S}(\mathbb{H}^d)$ in the sense that for any k, a constant C_k and an integer M_k exist such that

$$\|f \star g\|_{k,\mathcal{S}(\mathbb{H}^d)} \leq C_k \|f\|_{M_k, L^1_{\mathcal{S}}(\mathbb{H}^d)} \|g\|_{M_k,\mathcal{S}(\mathbb{H}^d)}.$$

Proof. Let us first observe that using Proposition 1.1.5, we have $\Delta^\ell_{\mathbb{H}}(f \star g) = f \star \Delta^\ell_H g$. Then we have to study the action of $M^k_{\mathbb{H}}$ on the convolution. This is described by the following lemma.

Lemma 1.3.1. *Let us define*

$$(M^\pm_j f)(Y,s) \stackrel{\text{def}}{=} (y_j \pm i\eta_j) f(Y,s) \quad \textit{and} \quad M_0 f(s) \stackrel{\text{def}}{=} -isf(Y,s).$$

Then we have the identities

$$\begin{aligned} M^\pm_j(f \star g) &= (M^\pm_j f) \star g + f \star (M^\pm_j g) \quad \textit{and} \\ M_0(f \star g) &= (M_0 f) \star g + f \star (M_0 g) - \sum_{j=1}^d \big((M^+_j f) \star (M^-_j g) - (M^-_j f) \star (M^+_j g) \big). \end{aligned}$$

Proof. Checking the first formula is obvious. As for the second one, we use that by definition of the convolution, we have

$$-is(f \star g)(w) = -is \int_{\mathbb{H}^d} f\big((Y-Y', s-s'-2\sigma(Y,Y')\big) g(Y',s')\, dw'.$$

From $-is = -i\big(s-s'-2\sigma(Y,Y')\big) - is' - 2i\sigma(Y,Y')$, we infer that

$$\begin{aligned} -is(f\star g)(w) &= \int_{\mathbb{H}^d} (M_0 f)\big((Y-Y', s-s'-2\sigma(Y,Y')\big) g(Y',s')\, dw' \\ &\qquad + \int_{\mathbb{H}^d} f\big((Y-Y', s-s'-2\sigma(Y,Y')\big) (M_0 g)(Y',s')\, dw' \\ &\qquad\quad - \int_{\mathbb{H}^d} 2i\sigma(Y,Y') f\big((Y-Y', s-s'-2\sigma(Y,Y')\big) g(Y',s')\, dw' \\ &= ((M_0 f)\star g)(w) + (f \star (M_0 g))(w) \\ &\qquad\quad - \int_{\mathbb{H}^d} 2i\sigma(Y,Y') f\big((Y-Y', s-s'-2\sigma(Y,Y')\big) g(Y',s')\, dw'\,. \end{aligned}$$

The symplectic form σ satisfies $\sigma(Y',Y')=0$. Thus by definition of σ

$$\begin{aligned}
\Delta(f,g)(w) &\stackrel{\text{def}}{=} \int_{\mathbb{H}^d} 2i\sigma(Y,Y')f\big((Y-Y',s-s'-2\sigma(Y,Y')\big)g(Y',s')\,dw'\\
&= \int_{\mathbb{H}^d} 2i\sigma(Y-Y',Y')f\big((Y-Y',s-s'-2\sigma(Y,Y')\big)g(Y',s')\,dw'\\
&= \sum_{j=1}^{d}\int_{\mathbb{H}^d}\Big(2i(\eta_j-\eta'_j)f\big(Y-Y',s-s'-2\sigma(Y,Y')\big)y'_jg(Y',s')\\
&\qquad\qquad -(y_j-y'_j)f\big(Y-Y',s-s'-2\sigma(Y,Y')\big)2i\eta'_jg(Y',s')\Big)\,dw'
\end{aligned}$$

By definition of the operator $M_j^{\pm}$, this gives

$$\begin{aligned}
\Delta_j(f,g)(w) &\stackrel{\text{def}}{=} \int_{\mathbb{H}^d}\Big(2i(\eta_j-\eta'_j)f\big(Y-Y',s-s'-2\sigma(Y,Y')\big)y'_jg(Y',s')\\
&\qquad\qquad -(y_j-y'_j)f\big(Y-Y',s-s'-2\sigma(Y,Y')\big)2i\eta'_jg(Y',s')\Big)\,dw'\\
&= \frac{1}{2}\big((M_j^+-M_j^-)f\big)\star\big(M_j^++M_j^-)g\big)-\frac{1}{2}\big((M_j^++M_j^-)f\big)\star\big(M_j^+-M_j^-)g\big)\\
&= (M_j^+f)\star(M_j^-g)-(M_j^-f)\star(M_j^+g).
\end{aligned}$$

This gives the result. $\square$

Conclusion of the proof of Proposition 1.3.2 If we denote, for α in $\mathbb{N}^{2d+1}$,

$$\mathcal{M}^\alpha \stackrel{\text{def}}{=} M_0^{\alpha_0}(M_1^+)^{\alpha_1}\cdot(M_d^+)^{\alpha_d}(M_1^-)^{\alpha_{d+1}}\cdots(M_1^-)^{\alpha_{2d}}$$

an iterated application of Lemma 1.3.1 implies that

$$M_{\mathbb{H}}^k\Delta_{\mathbb{H}}^k(f\star g)=\sum_{\widetilde{|}\alpha|+\widetilde{|}\alpha'|\leq 2k} a_{\alpha,\alpha'}(\mathcal{M}^\alpha f)\star(\mathcal{M}^{\alpha'}\Delta_{\mathbb{H}}^k g).$$

Young's inequality implies that

$$\begin{aligned}
\big\|M_{\mathbb{H}}^k\Delta_{\mathbb{H}}^\ell(f\star g)\big\|_{L^2} &\leq C_k\sum_{\widetilde{|}\alpha|+\widetilde{|}\alpha'|\leq 2k}\|\mathcal{M}^\alpha f\|_{L^1}\|\mathcal{M}^{\alpha'}\Delta_{\mathbb{H}}^k g\|_{L^2}\\
&\leq C_k\|f\|_{k,\mathcal{S}(\mathbb{H}^d)}\|g\|_{k,\mathcal{S}(\mathbb{H}^d)}.
\end{aligned}$$

This proves the proposition. $\square$

Chapter 2

The Schrödinger representation

Introduction

The purpose of this chapter is the detailled study of the Schrödinger representation of the Heisenberg group. Let us motivate this study. As already explained, the goal is this series of lectures is the definition of a Fourier transform on the Heisenberg group. In the case of a commutative group G, the Fourier transform is defined as a function on the dual goup $\widehat{G}$ of G which is the set of characters, i.e. the set of homomorphism of the group G into a group of complex number of modulus 1. In the case of $\mathbb{R}^n$, this space is parametrized by the dual space $(\mathbb{R}^n)^\star$ of $\mathbb{R}^n$ by the map

$$\left\{\begin{array}{ccc} (\mathbb{R}^n)^\star & \longrightarrow & \widehat{\mathbb{R}}^n \\ \xi & \longmapsto & x \mapsto e^{i\langle \xi, x\rangle}. \end{array}\right.$$

In the case of non commutative group, characters cannot be used anymore. For instance, for the Heisenberg group, it is possible to prove that the characters are exactly the ones of $T^\star\mathbb{R}^d \sim \mathbb{R}^{2d}$. Thus if we follow the classical way, we simply define for Fourier transform on $\mathbb{R}^{2d}$ of the mean value of the function with respect to the vertical variable. And thus, no inversion formula can be expected.

One of the fundamental idea of group theory is to use (family of) representations instead of characters. A representation is a homomorphism from the group to a subgroup of linear transform on a vector space. In the case of the Heisenberg group, among other possibilities, in particular the so called Bargman representation, we choose the Schrödinger reprensation. which is defined by

$$U^\lambda : \left\{\begin{array}{ccl} \mathbb{H}^d & \longrightarrow & \mathcal{U}(L^2(\mathbb{R}^d)) \\ w & \longmapsto & U^\lambda_w \;/\; U^\lambda_w \phi \stackrel{\text{def}}{=} e^{-i\lambda(s+2\langle \eta, \cdot - y\rangle)}\phi(\cdot - 2y). \end{array}\right.$$

As already mentionned, the motivation of this series of lectures in the definition and the study of the Fourier transform for integrable functions on the Heisenberg group. The family $(U^\lambda)_{\lambda\in\mathbb{R}\setminus\{0\}}$ is a candidate for substituting to the characters. Then for any integrable function f on $\mathbb{H}^d$, we can define

$$\mathcal{F}^{\mathbb{H}}(f)(\lambda) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w) U^\lambda_w \, dw$$

which is a bounded operator on $L^2(\mathbb{R}^d)$. Now let us think about the properties we expect for the Fourier transform:

– the Fourier transform of the convolution product of two integrable functions is the product of the Fourier transform (the property is purely algebraic);

– there is an inversion formula and a Fourier-Plancherel inequality (this can be proved in the frame work of operators defined here)

– the Fourier transform provides a spectral representation of the Laplace operator; for instance in the case of $\mathbb{R}^n$

$$(-\Delta f|f)_{L^2} = (2\pi)^{-d} \int_{(\mathbb{R}^n)^\star} |\xi|^2 \widehat{f}(\xi)\overline{\widehat{f}}(\xi) d\xi.$$

– and a principle claims that regalarity of the function impies deacy of the Fourier trnasform and the decay of the function implies "regularity" of the Fourier transform provided this make sense.

The last two point are highly related to the formula

$$|x|^2 e^{i\langle \xi, x\rangle} = -\Delta_\xi e^{i\langle \xi, x\rangle} \quad \text{and} \quad -\Delta_x e^{i\xi, x\rangle} = |\xi|^2 e^{i\langle \xi, x\rangle}$$

and demands to defined the Fourier transform as a function on the space of "frequencies" of $\mathbb{H}^d$. In other terms, this means to find a type of parametrization of the family $(U^\lambda)_{\lambda\in\mathbb{R}\setminus\{0\}}$.

In the first section, we are going to define the parametrization of the family $(U^\lambda)_{\lambda\in\mathbb{R}\setminus\{0\}}$ by considering its matrices in the basis of rescaled Hermite functions. This will leads us to the definition of the map

$$\begin{cases} (\mathbb{N}^{2d} \times \mathbb{R}\setminus\{0\}) \times T^\star\mathbb{R}^d & \longrightarrow \quad \mathbb{C} \\ \qquad (\widehat{w}, Y) & \longmapsto \quad \overline{(U_w^\lambda H_{m,\lambda}|H_{n,\lambda})_{L^2}} \stackrel{\text{def}}{=} e^{is\lambda}\mathcal{W}(n,m,\lambda,Y) \end{cases}$$

where $(H_{n,\lambda})_{n\in\mathbb{N}^d}$ denotes the family of rescaled Hermite functions (see forthcoming Definitions (2.6) and (2.14).

The second section is devoted to the study of this function $\mathcal{W}$. In particular we want to get some analogies with the two fondamental properties

$$|x|^2 e^{i\langle \xi, x\rangle} = -\Delta_\xi e^{i\langle \xi, x\rangle} \quad \text{and} \quad -\Delta_x e^{i\xi, x\rangle} = |\xi|^2 e^{i\langle \xi, x\rangle}.$$

2.1 The Schrödinger representation; definition and basis properties

We use the Schrödinger representation which is an irreducible representation of the Heisenberg group. We shall pay special attention to the behavior of that representation when the parameter λ (see below) tends to 0.

Definition 2.1.1. *For $w = (y, \eta, s)$ in $\mathbb{H}^d$, λ in $\mathbb{R}\setminus\{0\}$ and $u : \mathbb{R}^d \to \mathbb{R}$, let us define*

$$U_w^\lambda u(x) \stackrel{\text{def}}{=} e^{-i\lambda(s+2\langle \eta, x-y\rangle)} u(x-2y).$$

The basics property of U_w^λ, called Schrödinger representation, are described in the following proposition.

Proposition 2.1.1. *For any λ in $\mathbb{R}\setminus\{0\}$, the map*

$$U^\lambda : \begin{cases} \mathbb{H}^d & \longrightarrow \quad \mathcal{U}(L^2(\mathbb{R}^d)) \\ w & \longmapsto \quad U^\lambda_w \end{cases}$$

is a group homomorphism between $\mathbb{H}^d$ and the unitary group $\mathcal{U}(L^2(\mathbb{R}^d))$ of $L^2(\mathbb{R}^d)$.

Proof. The fact that U^λ_w is unitary in $L^2(\mathbb{R}^d)$ is clear. In order to show that U^λ is a homomorphism, we perform the following computations that rely on the definition of U^λ_w:

$$\begin{aligned} (U^\lambda_w U^\lambda_{w'} u)(x) &= U^\lambda_w \Big(e^{-i\lambda(s'+2\langle \eta', \cdot - y'\rangle)} u(\cdot - 2y') \Big)(x) \\ &= e^{-i\lambda(s+2\langle \eta, x-y\rangle)} e^{-i\lambda(s'+2\langle \eta', x-y'-2y\rangle)} u(x-2y'-2y) \\ &= e^{-i\lambda(s+s'+2\langle \eta, y'\rangle - 2\langle \eta', y\rangle + 2\langle \eta+\eta', x-(y+y')\rangle)} u(x-2(y'+y)). \end{aligned}$$

Remembering the definition of the product law on the Heisenberg group, we thus conclude that (with obvious notation)

$$(U^\lambda_w U^\lambda_{w'} \phi)(x) = e^{-i\lambda((w\cdot w')_s + 2\langle (w\cdot w')_\eta, x-(w\cdot w')_y\rangle)} \phi(x-2(w\cdot w')_y),$$

which completes the proof of the proposition. $\square$

We want to study the effect of the Laplace operator $\Delta_{\mathbb{H}}$ on $U^\lambda_w \phi$ for a C^2 real function ϕ on $\mathbb{R}^d$.

Proposition 2.1.2. *Let us define for u in $\mathcal{S}'(\mathbb{R}^d)$,*

$$\Delta^\lambda_{\rm osc} u(x) \stackrel{\rm def}{=} \sum_{j=1}^d \partial_j^2 u(x) - \lambda^2 |x|^2 u(x).$$

Let us also introduce the right-invariant *vector fields on the Heisenberg group and the associated Laplacian*

$$\widetilde{\mathcal{X}}_j \stackrel{\rm def}{=} \partial_{y_j} - 2\eta_j \partial_s\,, \quad \widetilde{\Xi}_j \stackrel{\rm def}{=} \partial_{\eta_j} + 2y_j\partial_s \quad \text{and} \quad \widetilde{\Delta}_{\mathbb{H}} \stackrel{\rm def}{=} \sum_{j=1}^d (\widetilde{\mathcal{X}}_j^2 + \widetilde{\Xi}_j^2).$$

Then, for any function C^2 function ϕ on $\mathbb{R}^d$, we have

$$\Delta_{\mathbb{H}} U^\lambda_w(\phi) = 4U^\lambda_w \Delta^\lambda_{osc}\phi \quad \text{and} \quad \widetilde{\Delta}_{\mathbb{H}} U^\lambda_w(\phi) = 4\Delta^\lambda_{\rm osc} U^\lambda_w(\phi).$$

Proof. Let us recall that

$$\mathcal{X}_j = \partial_{y_j} + 2\eta_j\partial_s\,, \quad \Xi_j = \partial_{\eta_j} - 2y_j\partial_s \quad \text{and} \quad \Delta_{\mathbb{H}} = \sum_{j=1}^d (\mathcal{X}_j^2 + \Xi_j^2).$$

As we have

$$\mathcal{X}_j\big(s+2\langle \eta, x-y\rangle\big) = 0 \quad \text{and} \quad \Xi_j\big(s+2\langle \eta, x-y\rangle\big) = -2(x_j-2y_j)$$

we discover that

$$\begin{aligned} \mathcal{X}_j(U^\lambda_w \phi(x)) &= -2e^{-i\lambda s - 2i\lambda\langle \eta, x-y\rangle} \partial_j \phi(x-2y) \quad \text{and} \\ \Xi_j(U^\lambda_w \phi(x)) &= -2i\lambda e^{-i\lambda s - 2i\lambda\langle \eta, x-y\rangle})(x_j-2y_j)\phi(x-2y) \end{aligned}$$

which can be written

$$\mathcal{X}_j U_w^\lambda \phi(x) = 2U_w^\lambda(\partial_j\phi)(x) \quad \text{and} \quad \mathcal{F}^{\mathbb{H}}(\Xi_j f)(\lambda) = 2i\lambda U_w^\lambda(M_j\phi)(x) \tag{2.1}$$

where M_j stands for the multiplication operator by x_j:

$$M_j u(x) \stackrel{\text{def}}{=} x_j u(x).$$

By an iteration we get

$$(\mathcal{X}_j^2 + \Xi_j^2)U_w^\lambda\phi(x) =)\big(U_w^\lambda(\partial_j^2 - \lambda^2 M_j\phi)\big)(x). \tag{2.2}$$

By summation, this gives Now because

$$\widetilde{\mathcal{X}}_j\big(s + 2\langle\eta, x-y\rangle\big) = -4\eta_j \quad \text{and} \quad \widetilde{\Xi}_j\big(s + 2\langle\eta, x-y\rangle\big) = 2x_j$$

we infer that

$$\widetilde{\mathcal{X}}_j U_w^\lambda\phi(x) = 2\partial_{x_j} U_w^\lambda\phi(x) \quad \text{and} \quad \widetilde{\Xi}_j U_w^\lambda\phi(x) = 2i\lambda M_j U_w^\lambda\phi(x). \tag{2.3}$$

By an iteration we get

$$(\widetilde{\mathcal{X}}_j^2 + \widetilde{\Xi}_j^2)U_w^\lambda\phi(x) = (\partial_j^2 - \lambda^2 M_j)U_w^\lambda\phi(x). \tag{2.4}$$

By summation, Identities (2.2) and (2.4) imply the result. □

This suggests to introduced the family of Hermite functions. Let us first define the following creation and annihilation operators on $\mathbb{R}^d$:

$$C_j \stackrel{\text{def}}{=} -\partial_j + M_j \quad \text{and} \quad A_j \stackrel{\text{def}}{=} \partial_j + M_j. \tag{2.5}$$

Let us define the family $(H_n)_{n\in\mathbb{N}^d}$ of the *Hermite functions* on $\mathbb{R}^d$ by

$$H_n \stackrel{\text{def}}{=} \Big(\frac{1}{2^{|n|}n!}\Big)^{\frac{1}{2}} C^n H_0 \quad \text{with} \quad C^n \stackrel{\text{def}}{=} \prod_{j=1}^d C_j^{n_j} \quad \text{and} \quad H_0(x) \stackrel{\text{def}}{=} \pi^{-\frac{d}{4}} e^{-\frac{|x|^2}{2}}, \tag{2.6}$$

with, as usual, $n! \stackrel{\text{def}}{=} n_1! \cdots n_d!$ and $|n| \stackrel{\text{def}}{=} n_1 + \cdots + n_d$.

Let us point out that the above normalization ensures the family $(H_n)_{n\in\mathbb{N}^d}$ to be an orthonormal basis of $L^2(\mathbb{R}^d)$.

In order to derive more important properties of Hermite functions, let us observe that

$$C_jA_j + \mathrm{Id} = -\partial_j^2 + M_j^2 \quad \text{and} \quad [C_j, A_j] = -2\,\mathrm{Id}\,. \tag{2.7}$$

This implies that

$$[-\partial_j^2 + M_j^2, C_j] = 2C_j \quad \text{and} \quad [\partial_j^2 - M_j^2, C_j] = 2A_j. \tag{2.8}$$

Once noticed that

$$(-\partial_j^2 + M_j^2)H_0 = H_0,$$

we get by an omitted induction on the length of n that

$$(-\partial_j^2 + M_j^2)H_n = (2n_j + 1)H_n \quad \text{and thus} \quad -\Delta^1_{\text{osc}} H_n = (2|n| + d)H_n. \tag{2.9}$$

From (2.6), (2.7) and the fact that $A_jH_0 \equiv 0$, it is obvious that we have

$$A_jH_n = \sqrt{2n_j}\, H_{n-\delta_j} \quad \text{and} \quad C_jH_n = \sqrt{2n_j+2}\, H_{n+\delta_j}, \tag{2.10}$$

where $\pm\delta_j$ is the shift of the j-th entry of the multi-index n by ± 1. Relations (2.5) can be written $2M_j = C_j + A_j$ and $2\partial_j = A_j - C_j$; this gives

$$\begin{aligned} M_jH_n &= \frac{1}{2}\big(\sqrt{2n_j+2}\, H_{n+\delta_j} + \sqrt{2n_j}H_{n-\delta_j}\big) \quad \text{and} \\ \partial_jH_n &= \frac{1}{2}\big(\sqrt{2n_j}\, H_{n-\delta_j} - \sqrt{2n_j+2}H_{n+\delta_j}\big). \end{aligned} \tag{2.11}$$

The following lemma will be useful in that follows. Its inequality explain that multiplying

Lemma 2.1.1. *Let us work in dimension* 1. *We have, for any positive integer* ℓ,

$$\begin{aligned} \Big(\frac{A\pm C}{2}\Big)^{\ell} H_n &= \sum_{\ell'=0}^{\ell} F^{\pm}_{\ell,\ell'}(n) H_{n+\ell-2\ell'} \quad \textit{with} \\ F^{\pm}_{\ell,\ell'}(n) &= \sum_{A\in\mathcal{P}(\ell,\ell')} (-1)^{k_A} \prod_{k=1}^{\ell} \sqrt{(2n+2N^{\pm}_{A,k})}. \end{aligned} \tag{2.12}$$

where $\mathcal{P}(\ell,\ell')$ *denotes the set of subsets of* $\{1,\cdots,\ell\}$ *with* ℓ' *elements and* k_A *are and* $N_{A,k}{}^{\pm}$ *are integers between* $-\ell$ *and* ℓ *such that* $n+N^{\pm}_A$ *is non negative. We also we have that*

$$\|(A\pm C)^{\ell}H_n\|_{L^2} \leq C_\ell (n+1)^{\frac{\ell}{2}}. \tag{2.13}$$

Proof. Let us prove this lemma by induction. In the case when ℓ equal to 1, by definition of A and C, we have

$$(A\pm C)H_n = \sqrt{2n}H_{n-1} \pm \sqrt{2n+2}H_{n+1}$$

and (2.12) is obviously satisfied for $\ell = 1$. Let us assume (2.12) with ℓ. Then using that

$$\begin{aligned} \Big(\frac{A\pm C}{2}\Big)^{\ell+1} H_n &= \sum_{\ell'=0}^{\ell} F^{\pm}_{\ell,\ell'}(n)\big(\sqrt{2n+2\ell-4\ell'}\, H_{n+\ell-2\ell'-1} \\ &\qquad\qquad \pm \sqrt{2n+2\ell-4\ell'+2}\, H_{n+\ell-2\ell'+1}\big) \\ &= \sum_{\ell'=0}^{\ell} F^{\pm}_{\ell,\ell'}(n)\big(\sqrt{2n+2\ell-4\ell'}\, H_{n+\ell+1-2\ell'-2} \\ &\qquad\qquad \pm \sqrt{2n+2\ell-4\ell'+2}\, H_{n+\ell+1-2\ell'}\big). \end{aligned}$$

Changing $\ell'' = \ell' + 1$ in the first sum gives

$$\begin{aligned} \Big(\frac{A\pm C}{2}\Big)^{\ell+1} H_n &= \sum_{\ell'=0}^{\ell} F^{\pm}_{\ell+1,\ell'}(n) H_{n+\ell+1-2\ell'} \quad \text{with} \\ F^{\pm}_{\ell+1,\ell'}(n) &\stackrel{\text{def}}{=} \sqrt{2n+2\ell-4\ell'}\,\big(F^{\pm}_{\ell,\ell'-1}(n) \pm F^{\pm}_{\ell,\ell'}(n)\big). \end{aligned}$$

This proves (2.12) for $\ell + 1$.

Let us prove the inequality. The identity (2.12) together with the fact that $(H_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$ implies that

$$\begin{aligned}\|(A\pm C)^\ell H_n\|_{L^2}^2 &= \sum_{0\leq\ell'\leq\ell}\big(F^{\pm}_{\ell,\ell'}(n)\big)^2\\ &\leq C_\ell(n+1)^{\frac{\ell}{2}}.\end{aligned}$$

This proves the lemma. □

Let us introduce the following *rescaled Hermite functions* :

$$H_{n,\lambda}(x)\stackrel{\mathrm{def}}{=}|\lambda|^{\frac{d}{4}}H_n(|\lambda|^{\frac12}x)\quad\text{which satisfy}\quad -\Delta^\lambda_{\mathrm{osc}}H_{n,\lambda}=|\lambda|(2|n|+d)H_{n,\lambda}. \tag{2.14}$$

Now let us consider, for a fixed positive λ, the matrix of the operator U^λ_w in the orthogonal basis of $L^2(\mathbb{R}^d)$ given by $(H_{n,\lambda})_{n\in\mathbb{N}}$. Its coefficient are

$$(U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2}.$$

This function will play a role which is analog to $e^{-i\langle\xi,x\rangle}$ in the case of $\mathbb{R}^n$.

2.2 Oscillatory functions associated with the Schrödinger representation

The purpose of this section is the determination of the extension of the function $\mathcal{W}$ (still denoted by $\mathcal{W}$) on the whole space $\widehat{\mathbb{H}}^d$. Let us define precisely the function which will play a crucial role in that follows.

Proposition 2.2.1. *Let us denote by $\widetilde{\mathbb{H}}^d$ the set $\mathbb{N}^{2d}\times\mathbb{R}\setminus\{0\}$. We denote by $\widehat{w}=(n,m,\lambda)$ a generic element of $\widetilde{\mathbb{H}}^d$. Then we define*

$$\mathcal{W}\quad\left\{\begin{array}{ccl}\widetilde{\mathbb{H}}^d\times T^\star\mathbb{R}^d & \longrightarrow & \mathbb{C}\\ (\widehat{w},Y) & \longmapsto & \displaystyle\int_{\mathbb{R}^d} e^{2i\lambda\langle\eta,z\rangle}H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)dz\end{array}\right. \tag{2.15}$$

We have that

$$(U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2}=e^{-is\lambda}\overline{\mathcal{W}}(\widehat{w},Y).$$

Proof. Changing variables $z=x-y$, we gather that

$$\begin{aligned}(U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2} &= e^{-is\lambda}\int_{\mathbb{R}^d} e^{-2i\lambda\langle\eta,x-y\rangle}|\lambda|^{\frac{d}{2}}H_m\big(|\lambda|^{\frac12}(x-2y)\big)H_n(|\lambda|^{\frac12}x)\,dx\\ &= e^{-is\lambda}\int_{\mathbb{R}^d} e^{-2i\lambda\langle\eta,x-y\rangle}H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)\,dz.\end{aligned}$$

This proves the proposition. □

Let us first see the consequence on the function $\mathcal{W}$ of the fact that, fr any λ, the map U^λ is a homomorphism from $\mathbb{H}^d$ into the unitary operators on $L^2(\mathbb{R}^d)$.

Proposition 2.2.2. *For any (n,m,λ,Y) in $\widetilde{\mathbb{H}}^d \times T^\star\mathbb{R}^d$, we have*

$$\sum_{n\in\mathbb{N}^d} |\mathcal{W}(n,m,\lambda,Y)|^2 = \sum_{m\in\mathbb{N}^d} |\mathcal{W}(n,m,\lambda,Y)|^2 = 1. \tag{2.16}$$

Moreover, we have, for any $\widehat{w}$ in $\widetilde{\mathbb{H}}^d$ and any couple (Y_1,Y_2) of $(T^\star\mathbb{R}^d)^2$,

$$\mathcal{W}(n,m,\lambda,Y_1+Y_2) = e^{2i\lambda\sigma(Y_1,Y_2)} \sum_{\ell\in\mathbb{N}^d} \mathcal{W}(n,\ell,\lambda,Y_1)\mathcal{W}(m,\ell,\lambda,Y_2) \tag{2.17}$$

Proof. The fact that, for any m in $\mathbb{N}^d$ and any λ in $\mathbb{R}\setminus\{0\}$, the L^2 norm of $U^\lambda_w H_{m,\lambda}$ is 1 implies that (2.16). Moreover, we have

$$\big(U^\lambda_{w_1}U^\lambda_{w_2}H_{m,\lambda}|H_{n,\lambda}\big)_{L^2} = e^{-i(s_1+s_2)\lambda}\sum_{\ell\in\mathbb{N}^d} \overline{\mathcal{W}}(m,\ell,\lambda,Y_1)\overline{\mathcal{W}}(\ell,n,\lambda,Y_2).$$

As $U^\lambda_{w_1}U^\lambda_{w_2} = U^\lambda_{w_1\cdot w_2}$, we infer that

$$\big(U^\lambda_{w_1}U^\lambda_{w_2}H_{m,\lambda}|H_{n,\lambda}\big)_{L^2} = e^{-i\lambda(s_1+s_2-2\sigma(Y_1,Y_2))}\overline{\mathcal{W}}(n,m,Y_1+Y_2).$$

This conclude the proof of the proposition. □

Remark 2.2.1. *Formula (2.17) is the analog here of the formula $e^{i\langle\xi,x_1+x_2\rangle} = e^{i\langle\xi,x_1\rangle}e^{i\langle\xi,x_2\rangle}$.*

From now, we want to the function $e^{is\lambda}\mathcal{W}(\widehat{w},Y)$ plays the role on the function $e^{i\langle\xi,x\rangle}$. The duality between $\mathbb{R}^d$ and $(\mathbb{R}^d)^\star$ can be interpret by the two relations

$$\begin{aligned} -\Delta_x e^{i\xi,x\rangle} &= |\xi|^2 e^{i\langle\xi,x\rangle} \quad \text{and} &(2.18)\\ |x|^2 e^{i\langle\xi,x\rangle} &= -\Delta_\xi e^{i\langle\xi,x\rangle}. &(2.19)\end{aligned}$$

We want some analog to this two relations. These two relations are the key point for the exchange between the decay and regularity through the Fourier transform. The analog of the first relation (2.18) is almost already established.

Proposition 2.2.3. *For any $(\widehat{w},Y)$ in $\widetilde{\mathbb{H}}^d \times T^\star\mathbb{R}^d$, we have*

$$\begin{aligned} -\Delta_{\mathbb{H}}\big(e^{is\lambda}\mathcal{W}(\widehat{w},Y)\big) &= 4|\lambda|(2|m|+d)e^{is\lambda}\mathcal{W}(\widehat{w},Y) \quad \textit{and}\\ -\widetilde{\Delta}_{\mathbb{H}}\big(e^{is\lambda}\mathcal{W}(\widehat{w},Y)\big) &= 4|\lambda|(2|n|+d)e^{is\lambda}\mathcal{W}(\widehat{w},Y).\end{aligned}$$

Proof. The conclusion of Proposition 2.2.1 can be written

$$e^{is\lambda}\mathcal{W}(\widehat{w},Y) = \overline{(U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2}}.$$

Then let us write that Proposition 2.1.2 and Relations implies (2.14) imply that

$$\begin{aligned} -(\mathcal{X}_j^2+\Xi_j^2)(U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2} &= (-(\mathcal{X}_j^2+\Xi_j^2)U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2}\\ &= 4|\lambda|(2m_j+1)(U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2}. &(2.20)\end{aligned}$$

Along the same lines, we have

$$\begin{aligned} -(\widetilde{\mathcal{X}}_j^2+\widetilde{\Xi}_j^2)(U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2} &= (-(\widetilde{\mathcal{X}}_j^2+\widetilde{\Xi}_j^2)U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2}\\ &= 4|\lambda|(2n_j+1)(U^\lambda_w H_{m,\lambda}|H_{n,\lambda})_{L^2}. &(2.21)\end{aligned}$$

This proves the result by summation over the index j. □

Let us establish another formula about the function $\mathcal{W}$ which will be useful in the future.

Proposition 2.2.4. *For any j in $\{1, \cdots, d\}$, we have, for any $(n, m, \lambda, Y) \in \widetilde{\mathbb{H}}^d \times T^\star\mathbb{R}^d$,*

$$i\lambda R_j \mathcal{W}(n, m, \lambda, Y) = |\lambda|(m_j - n_j)\mathcal{W}(n, m, \lambda) \quad \textit{with} \quad R_j \stackrel{\text{def}}{=} y_j \partial_{\eta_j} - \eta_j \partial_{y_j}.$$

Proof. From the identities

$$\widetilde{\mathcal{X}}_j^2 - \mathcal{X}_j^2 = -8\eta_j \partial_{y_j} \partial_s \quad \text{and} \quad \widetilde{\Xi}_j^2 - \Xi_j^2 = 8y_j \partial_{\eta_j} \partial_s,$$

we infer that

$$\widetilde{\mathcal{X}}_j^2 + \widetilde{\Xi}_j^2 - \mathcal{X}_j^2 - \Xi_j^2 = 8R_j \partial_s.$$

Thus we deduce the result from (2.20) and (2.21). □

This proposition has an important corollary.

Corollary 2.2.1. *For any integer k, a constant C exists such that*

$$|\mathcal{W}(n, m, \lambda, Y)| \leq C_k \biggl(\frac{\langle Y \rangle \bigl(1 + \sqrt{|\lambda|(|n + m| + d)}\bigr)}{1 + |m - n|}\biggr)^k \quad \textit{with} \quad \langle Y \rangle \stackrel{\text{def}}{=} \sqrt{1 + |Y|^2}\,.$$

Proof. By an iteration of the identity of Proposition 2.2.4, we get that, for any k,

$$|m - n|^k |\mathcal{W}(n, m, \lambda, Y)| \leq C_k \langle Y \rangle^k \sup_{|\alpha| \leq k} |\partial_Y^\alpha \mathcal{W}(n, m, Y)|. \tag{2.22}$$

Then, let us established the following lemma about derivatives of $\mathcal{W}$ with respect to Y.

Lemma 2.2.1. *For any α in $\mathbb{N}^{2d}$, a constant C_α exists that for*

$$\forall (\widehat{w}, Y) \in \widetilde{\mathbb{H}}^d \times T^\star\mathbb{R}^d\,, \ |\partial_Y^\alpha \mathcal{W}(n, m, \lambda, Y)| \leq C_\alpha \bigl(|\lambda|(|n + m| + d)\bigr)^{\frac{|\alpha|}{2}}.$$

Proof. By differentiation under the integral, we have, using Leibnitz formula,

$$\begin{aligned}
\partial_y^{\alpha_1} \partial_\eta^{\alpha_2} \mathcal{W}(\widehat{w}, Y) &= \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z\rangle} (i\mathrm{sg}\lambda)^{|\alpha_2|} |\lambda|^{\frac{|\alpha_2|}{2}} \\
&\qquad\qquad \times |\lambda|^{\frac{|\alpha_2|}{2}} (y + z - y + z)^{\alpha_2} \partial_y^{\alpha_1} \bigl(H_{n,\lambda}(y + z) H_{m,\lambda}(-y + z)\bigr)\, dz \\
&= (i\mathrm{sg}(\lambda))^{|\alpha_2|} |\lambda|^{\frac{|\alpha_1| + |\alpha_2|}{2}} \sum_{\substack{\beta_1 \leq \alpha_1 \\ \beta_2 \leq \alpha_2}} (-1)^{\beta_1} C_{\alpha_1}^{\beta_1} C_{\alpha_2}^{\beta_2} \\
&\qquad \times \int_{\mathbb{R}^d} e^{2i\lambda\langle \eta, z\rangle} (M^{\beta_2} \partial^{\beta_1} H_n)_\lambda (y + z) (M^{\beta_2 - \beta_2} \partial^{\alpha_1 - \beta_1} H_m)_\lambda (-y + z) dz.
\end{aligned}$$

Stating $\alpha = (\alpha_1, \alpha_2)$, we infer that

$$\bigl|\partial_Y^\alpha \mathcal{W}(\widehat{w}, Y)\bigr| \leq |\lambda|^{\frac{|\alpha_1| + |\alpha_2|}{2}} \sum_{\substack{\beta_1 \leq \alpha_1 \\ \beta_2 \leq \alpha_2}} C_{\alpha_1}^{\beta_1} C_{\alpha_2}^{\beta_2} \|M^{\beta_2} \partial^{\beta_1} H_n\|_{L^2} \|M^{\beta_2 - \beta_2} \partial^{\alpha_1 - \beta_1} H_m\|_{L^2}. \tag{2.23}$$

□

Conclusion of the proof of Corollay 2.2.1 Plugging the estimate of the above lemma in (2.22) gives, for any k

$$|m-n|^k|\mathcal{W}(n,m,\lambda,Y)| \leq C_k\langle Y\rangle\big(|\lambda|(|n+m|+d)\big)^{\frac{k}{2}}$$

and this concludes the proof. $\square$

Remark 2.2.2. *Let us point out that on subsets of $\widehat{\mathbb{H}}^d$ for which we have $|\lambda|(2|n|+d)\leq C$, then Corollary 2.2.1 provide decay with respect to $|m-n|$. This can be interpreted as the fact that on such set, the function $\mathcal{W}$ has arbitrary polynomial decay with respect to the distance to the diagonal set of $\mathbb{N}^d\times\mathbb{N}^d$.*

Now let us state a equivalent of Relation (2.19).

Lemma 2.2.2. *Let us define for a function θ on $\widetilde{\mathbb{H}}^d$, the operator $\widehat{\Delta}$ by*

$$\begin{aligned}\widehat{\Delta}\theta(n,m,\lambda) &= -\frac{1}{2|\lambda|}(|n+m|+d)\theta(n,m,\lambda)\\ &+\frac{1}{2|\lambda|}\sum_{j=1}^{d}\Big\{\sqrt{(n_j+1)(m_j+1)}\,\theta(n+\delta_j,m+\delta_j,\lambda)+\sqrt{n_jm_j}\,\theta(n-\delta_j,m-\delta_j,\lambda)\Big\}.\end{aligned} \tag{2.24}$$

The operator $\widehat{\Delta}$ is symmetric in the sense that for any couple of functions (θ_1,θ_2) on $\widetilde{\mathbb{H}}^d$ supported in $\widetilde{\mathbb{H}}^d_N \stackrel{def}{=} \big\{(n,m,\lambda\in\widetilde{\mathbb{H}}^d\,/|n+m|\leq N\big\}$, we have

$$\forall\lambda\in\mathbb{R}\setminus\{0\}\,,\ \big(\widehat{\Delta}\theta_1(\cdot,\lambda)\big|\theta_2(\cdot,\lambda)_{L^2(\mathbb{N}^{2d})}=\big(\theta_1(\cdot,\lambda)\big|\widehat{\Delta}\theta_2(\cdot,\lambda)\big)_{L^2(\mathbb{N}^{2d})}. \tag{2.25}$$

Moreover, we have

$$\forall(\widehat{w},Y)\in\widetilde{\mathbb{H}}^d\times T^\star\mathbb{R}^d\,,\ |Y|^2\mathcal{W}(\widehat{w},Y)=-\widehat{\Delta}\mathcal{W}(\widehat{w},Y). \tag{2.26}$$

Proof. Let us observe that by definition, for any couple of functions (θ_1,θ_2) on $\widetilde{\mathbb{H}}^d$ supported in $\widetilde{\mathbb{H}}^d_N$, we have, for any λ in $\mathbb{R}\setminus\{0\}$,

$$\begin{aligned}2|\lambda|(\widehat{\Delta}\theta_1(\cdot,\lambda)|\theta_2))_{L^2(\mathbb{N}^{2d})} &= -\sum_{(n,m)\in\mathbb{N}^{2d}}(|n+m|+d)\theta_1(n,m,\lambda)\overline{\theta}_2(n,m,\lambda)\\ &+\sum_{j=1}^{d}\sum_{(n,m)\in\mathbb{N}^{2d}}\sqrt{(n_j+1)(m_j+1)}\,\theta_1(n+\delta_j,m+\delta_j,\lambda)\overline{\theta}_2(n,m,\lambda)\\ &+\sum_{j=1}^{d}\sum_{(n,m)\in\mathbb{N}^{2d}}\sqrt{n_jm_j}\,\theta_1(n-\delta_j,m-\delta_j,\lambda)\overline{\theta}_2(n,m,\lambda)\,.\end{aligned}$$

Changing variables $(n',m')=(n+\delta_j,m+\delta_j)$ in the second sum and $(n',m')=(n-\delta_j,m-\delta_j)$ in the third gives

$$\begin{aligned}2|\lambda|(\widehat{\Delta}\theta_1(\cdot,\lambda)|\theta_2))_{L^2(\mathbb{N}^{2d})} &= -\sum_{(n,m)\in\mathbb{N}^{2d}}(|n+m|+d)\theta_1(n-\delta_j,m-\delta_j,\lambda)\overline{\theta}_2(n,m,\lambda)\\ &+\sum_{j=1}^{d}\sum_{(n,m)\in\mathbb{N}^{2d}}\sqrt{n_jm_j}\,\theta_1(n,m,\lambda)\overline{\theta}_2(n-\delta_j,m-\delta_j,\lambda)\\ &+\sum_{j=1}^{d}\sum_{(n,m)\in\mathbb{N}^{2d}}\sqrt{(n_j+1)(m_j+1)}\,\theta_1(n,m,\lambda)\overline{\theta}_2(n+\delta_j,m+\delta_j,\lambda)\,.\end{aligned}$$

This means exactly that $2|\lambda|(\widehat{\Delta}\theta_1(\cdot,\lambda)|\theta_2))_{L^2(\mathbb{N}^{2d})} = 2|\lambda|(\theta_1(\cdot,\lambda)|\widehat{\Delta}\theta_2))_{L^2(\mathbb{N}^{2d})}$ and (2.25) is proved.

Now let us compute $|Y^2|\mathcal{W}(\widehat{w},Y)$. As

$$|Y|^2 e^{-2i\lambda\langle\eta,z\rangle} = (|y|^2+|\eta|^2)e^{-2i\lambda\langle\eta,z\rangle} = \Big(|y|^2 - \frac{1}{4\lambda^2}\Delta_z\Big)\big(e^{-2i\lambda\langle\eta,z\rangle}\big),$$

we get by definition of $\mathcal{W}$ and after integrations by parts,

$$\begin{aligned}
|Y|^2\mathcal{W}(\widehat{w},Y) &= \int_{\mathbb{R}^d}\Big(|y|^2-\frac{1}{4\lambda^2}\Delta_z\Big)\big(e^{-2i\lambda\langle\eta,z\rangle}\big)H_{n,\lambda}(y+z)H_{m,\lambda}(-y+z)\,dz\\
&= \int_{\mathbb{R}^d} e^{-2i\lambda\langle\eta,z\rangle}|\lambda|^{\frac{d}{2}}\mathcal{I}(\widehat{w},y,z)\,dz \quad \text{with}\\
\mathcal{I}(\widehat{w},y,z) &\stackrel{\text{def}}{=} \Big(|y|^2-\frac{1}{4\lambda^2}\Delta_z\Big)\big(H_n(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z))\big).
\end{aligned}$$

Using Leibniz formula, the chain rule and that $4|y|^2 = |y+z|^2+|y-z|^2-2(y+z)\cdot(y-z)$, we get

$$\begin{aligned}
\mathcal{I}(\widehat{w},y,z) &= -\frac{1}{4\lambda^2}\big((\Delta_z-\lambda^2|y+z|^2)H_n(|\lambda|^{\frac12}(y+z))\big)H_m(|\lambda|^{\frac12}(-y+z))\\
&\quad -\frac{1}{4\lambda^2}\big((\Delta_z-\lambda^2|y-z|^2)H_m(|\lambda|^{\frac12}(-y+z))\big)H_n(|\lambda|^{\frac12}(y+z))\\
&\quad -\frac{1}{2|\lambda|}\sum_{j=1}^d(\partial_jH_n)(|\lambda|^{\frac12}(y+z))(\partial_jH_m)(|\lambda|^{\frac12}(-y+z))\\
&\quad -\frac12(z+y)\cdot(z-y)H_n(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z)).
\end{aligned}$$

As $H_{n,\lambda}$ are eigenfunctions of Δ^λ_{osc}, we infer that

$$\begin{aligned}
\mathcal{I}(\widehat{w},y,z) &= \frac{1}{2|\lambda|}(|n+m|+d)H_n(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z))\\
&\quad -\frac{1}{2|\lambda|}\sum_{j=1}^d\Big\{(\partial_jH_n)(|\lambda|^{\frac12}(y+z))(\partial_jH_m)(|\lambda|^{\frac12}(-y+z))\\
&\qquad\qquad + (M_jH_n)(|\lambda|^{\frac12}(y+z))(M_jH_m)(|\lambda|^{\frac12}(-y+z)).
\end{aligned}$$

Then using Relations (2.11) gives Equality (2.26). □

The equivalence for the multiplication by is is different. Let us observe that

$$\begin{aligned}
ise^{is\lambda}\mathcal{W}(\widetilde{w},Y) &= \Big(\frac{d}{d\lambda}e^{is\lambda}\Big)\mathcal{W}(\widehat{w},Y)\\
&= \frac{d}{d\lambda}\Big(e^{is\lambda}\mathcal{W}(\widehat{w},Y)\Big) - e^{is\lambda}\frac{d}{d\lambda}\mathcal{W}(\widehat{w},Y). \qquad (2.27)
\end{aligned}$$

The computation of the derivative of $\mathcal{W}$ is done in the proof of the following lemma.

Lemma 2.2.3. *For any fonction θ on $\widetilde{\mathbb{H}}^d$, let us define let us define*

$$\begin{aligned}
&\widehat{\underline{\mathcal{D}}}_\lambda\theta(n,m,\lambda) \stackrel{\text{def}}{=} -\frac{d}{2\lambda}\theta(n,m,\lambda))\\
&+\frac{1}{2\lambda}\sum_{j=1}^d\Big\{\sqrt{(n_j+1)(m_j+1)}\,\theta(n+\delta_j,m+\delta_j,Y)-\sqrt{n_jm_j}\,\mathcal{W}(n-\delta_j,m-\delta_j,Y)\Big\}. \qquad (2.28)
\end{aligned}$$

The operator a formal adjoint in the sense that for any couple of functions (θ_1, θ_2) on $\widetilde{\mathbb{H}}^d$ supported in $\widetilde{\mathbb{H}}^d_N \stackrel{\text{def}}{=} \big\{(n,m,\lambda \in \widetilde{\mathbb{H}}^d / |n+m| \leq N\big\}$, we have, for any λ in $\mathbb{R}\setminus\{0\}$,

$$
(\widehat{\underline{\mathcal{D}}}_\lambda \theta_1(\cdot,\lambda)|\theta_2(\cdot,\lambda))_{L^2(\mathbb{N}^{2d})} = -(\theta_1(\cdot,\lambda)|\widehat{\underline{\mathcal{D}}}_\lambda \theta_2(\cdot,\lambda))_{L^2(\mathbb{N}^{2d})} - \frac{d}{\lambda}(\theta_1(\cdot,\lambda)|\theta_2(\cdot,\lambda))_{L^2(\mathbb{N}^{2d})}. \tag{2.29}
$$

Morevor, we have

$$
\frac{d}{d\lambda}\mathcal{W}(\widehat{w},Y) = \widehat{\underline{\mathcal{D}}}_\lambda \mathcal{W}(\widehat{w},Y).
$$

Proof. By definition of the operator $\widehat{\underline{\mathcal{D}}}_\lambda$, we have, for any λ in $\mathbb{R}\setminus\{0\}$,

$$
\begin{aligned}
2\lambda(\widehat{\underline{\mathcal{D}}}_\lambda \theta_1(\cdot,\lambda)|\theta_2(\cdot,\lambda))_{L^2(\mathbb{N}^{2d})} &= -d \sum_{(n,m)\in\mathbb{N}^{2d}} \theta_1(n,m,\lambda)\overline{\theta}_2(n,m,\lambda)\\
+\sum_{j=1}^d \sum_{(n,m)\in\mathbb{N}^{2d}} &\sqrt{(n_j+1)(m_j+1)}\,\theta_1(n+\delta_j,m+\delta_j,\lambda)\overline{\theta}_2(n,m,\lambda)\\
&-\sum_{j=1}^d \sum_{(n,m)\in\mathbb{N}^{2d}} \sqrt{n_jm_j}\,\theta_1(n-\delta_j,m-\delta_j,\lambda)\overline{\theta}_2(n,m,\lambda)\,.
\end{aligned}
$$

Changing variables $(n',m') = (n+\delta_j, m+\delta_j)$ in the second sum and $(n',m') = (n-\delta_j, m-\delta_j)$ in the third gives

$$
\begin{aligned}
2\lambda(\widehat{\underline{\mathcal{D}}}_\lambda \theta_1(\cdot,\lambda)|\theta_2(\cdot,\lambda))_{L^2(\mathbb{N}^{2d})} &= -d \sum_{(n,m)\in\mathbb{N}^{2d}} \theta_1(n-\delta_j,m-\delta_j,\lambda)\overline{\theta}_2(n,m,\lambda)\\
&+\sum_{j=1}^d \sum_{(n,m)\in\mathbb{N}^{2d}} \sqrt{n_jm_j}\,\theta_1(n,m,\lambda)\overline{\theta}_2(n-\delta_j,m-\delta_j,\lambda)\\
-\sum_{j=1}^d \sum_{(n,m)\in\mathbb{N}^{2d}} &\sqrt{(n_j+1)(m_j+1)}\,\theta_1(n,m,\lambda)\overline{\theta}_2(n+\delta_j,m+\delta_j,\lambda)\,.
\end{aligned}
$$

This means exactly that

$$
2\lambda(\widehat{\underline{\mathcal{D}}}_\lambda \theta_1(\cdot,\lambda)|\theta_2(\cdot,\lambda))_{L^2(\mathbb{N}^{2d})} = 2d\lambda(\theta_1(\cdot,\lambda)|\theta_2(\cdot,\lambda))_{L^2(\mathbb{N}^{2d})} - 2\lambda(\widehat{\underline{\mathcal{D}}}_\lambda \theta_1(\cdot,\lambda)|\theta_2(\cdot,\lambda))_{L^2(\mathbb{N}^{2d})}
$$

and (2.29) is proved.

Now let us compute the derivative of $\mathcal{W}(\widehat{w},Y)$ with respect to λ. By definition of $\mathcal{W}$, and using the chain rule and the differentiation theorem for integrals, we get

$$
\begin{aligned}
\frac{d}{d\lambda}\mathcal{W}(\widehat{w},Y) &= \int_{\mathbb{R}^d} \frac{d}{d\lambda}\Big(|\lambda|^{\frac{d}{2}} e^{2i\lambda\langle\eta,z\rangle} H_n(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z))\Big)dz\\
&= \frac{d}{2\lambda}\mathcal{W}(n,m,\lambda,Y) + \mathcal{W}_1(n,m,\lambda,Y) + \mathcal{W}_2(n,m,\lambda,Y) \quad \text{with}\\
\mathcal{W}_1(\widehat{w},Y) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^d} 2i\langle\eta,z\rangle e^{2i\lambda\langle\eta,z\rangle}|\lambda|^{\frac{d}{2}} H_n(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z))\,dz \text{ and}\\
\mathcal{W}_2(\widehat{w},Y) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^d} e^{2i\lambda\langle\eta,z\rangle}|\lambda|^{\frac{d}{2}} \frac{d}{d\lambda}\big(H_n(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z))\big)\,dz.
\end{aligned}
$$

We have

$$
2i\langle\eta,z\rangle e^{2i\lambda\langle\eta,z\rangle} = \frac{1}{\lambda}R\,e^{2i\lambda\langle\eta,z\rangle},
$$

where R denote the radial vector field on $\mathbb{R}^d$. An integration by parts gives

$$
\begin{aligned}
\mathcal{W}_1(\widehat{w},Y) = & -\frac{d}{\lambda}\mathcal{W}(\widehat{w},Y) \\
& + \frac{|\lambda|^{\frac{d}{2}}}{\lambda}\int_{\mathbb{R}^d} e^{2i\lambda\langle\eta,z\rangle} R_z\big(H_n(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z))\big)dz.
\end{aligned}
\tag{2.30}
$$

Now let us compute

$$
\mathcal{J}(\widehat{w},y,z) \stackrel{\text{def}}{=} \Big(\frac{d}{d\lambda} - \frac{1}{\lambda}R_z\Big)\big(H_n(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z))\big).
$$

From the chain rule we get

$$
\begin{aligned}
\mathcal{J}(\widehat{w},y,z) = \frac{|\lambda|^{\frac12}}{2\lambda}\sum_{j=1}^d \Big\{ & (y_j+z_j)(\partial_j H_n)(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z)) \\
& + H_n(|\lambda|^{\frac12}(y+z))(-y_j+z_j)(\partial_j H_m)(|\lambda|^{\frac12}(-y+z)) \\
& - 2z_j(\partial_j H_n)(|\lambda|^{\frac12}(y+z))H_m(|\lambda|^{\frac12}(-y+z)) \\
& - 2H_n(|\lambda|^{\frac12}(y+z))z_j(\partial_j H_m)(|\lambda|^{\frac12}(-y+z))\Big\}\cdot
\end{aligned}
$$

This gives

$$
\begin{aligned}
\mathcal{J}(\widehat{w},y,z) = -\frac{1}{2\lambda}\sum_{j=1}^d \Big\{ & (\partial_j H_n)(|\lambda|^{\frac12}(y+z))|\lambda|^{\frac12}(-y_j+z_j)H_m(|\lambda|^{\frac12}(-y+z)) \\
& + |\lambda|^{\frac12}(y_j+z_j)H_n(|\lambda|^{\frac12}(y+z))(\partial_j H_m)(|\lambda|^{\frac12}(-y+z))\Big\}
\end{aligned}
$$

which writes

$$
\begin{aligned}
\mathcal{J}(\widehat{w},y,z) = -\frac{1}{2\lambda}\sum_{j=1}^d \Big\{ & (\partial_j H_n)(|\lambda|^{\frac12}(y+z))(M_j H_m)(|\lambda|^{\frac12}(-y+z)) \\
& + (M_j H_n)(|\lambda|^{\frac12}(y+z))(\partial_j H_m)(|\lambda|^{\frac12}(-y+z))\Big\}\cdot
\end{aligned}
$$

Using Relations (2.11) concludes the proof of the Lemma.

□